Role of Singular Tori in the Dynamics of Spatiotemporal Nonlinear Wave Systems

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We show that the peculiar topological properties inherent to singular tori play a major role in the spatiotemporal dynamics of counterpropagating nonlinear waves. Under rather general conditions, these Hamiltonian wave systems exhibit a relaxation process towards a stationary state. We show that this stationary state converges exponentially towards the singular torus of the associated Liouville-integrable Hamiltonian system in the limit of an infinite medium. The singular torus then appears as an attractor for the infinite dimensional dynamical system, a feature which is illustrated by several key models of spatiotemporal wave interactions.

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Hamiltonian integrable systems have a long history dating back to Liouville in the mid-nineteenth century. The modern mathematical foundations of this domain were established, in particular, by Arnold [1] one hundred years later. In these last decades, while the physics community of dynamical systems was mainly concerned with the theory of chaos, novel methods were introduced in mathematics to study integrable systems [2–4]. More recently, the relevance of these mathematical structures has been analyzed in the context of mechanical systems characterized by a small number of degrees of freedom [5–9]. This Letter is aimed at describing a novel class of infinite dimensional dynamical systems in which these novel mathematical concepts find a remarkable physical application.

We consider a Hamiltonian H with 2 degrees of freedom defined on the four-dimensional phase space $\mathbb{R}^4$. The Hamiltonian H is said to be Liouville integrable if it has as many independent constants of motion as number of degrees of freedom. Since H does not depend on time, it will be integrable if there exists a function K which Poisson commutes with H, i.e., $\{H, K\} = 0$. From our perspective, the essential object in the study of integrable systems is not the Hamiltonian H but the energy-momentum map $\mathcal{F}:(p_1, q_1, p_2, q_2) \to (H, K)$ [2] where (p_1, q_1, p_2, q_2) are coordinates of $\mathbb{R}^4$. The image of $\mathcal{F}$ is called the energy-momentum set or bifurcation diagram. The Liouville-Arnold theorem states that the inverse image $\mathcal{F}^{-1}(h, k)$ of a point $(H = h,\ K = k)$ of the energy-momentum set is a two-dimensional torus. This theorem assumes that this preimage is connected and compact, and that the two gradient vectors of the phase space $\vec{\nabla}H = (\frac{\partial H}{\partial p_1}, \frac{\partial H}{\partial q_1}, \frac{\partial H}{\partial p_2}, \frac{\partial H}{\partial q_2})$ and $\vec{\nabla}K$ are not parallel for any point of the torus. Under these conditions, the corresponding point (h, k) is a regular point of the image of $\mathcal{F}$. At a regular point, action-angle variables can be introduced [1]. However, if the two vectors $(\vec{\nabla}H, \vec{\nabla}K)$ are parallel for some points of the torus, then the preimage $\mathcal{F}^{-1}(h, k)$ is no longer a regular torus, but a singular torus. The topology of the singular torus can be of different types: a point (for an equilibrium), a circle (for a periodic orbit), or a pinched torus, to cite a few. A pinched torus is a regular torus whose radius has been pinched to zero in one point, as illustrated in Fig. 1(a). Simple integrable Hamiltonians with pinched tori may be constructed with the momentum $K = (p_1^2 + q_1^2 - p_2^2 - q_2^2)/2$, which corresponds to a $1{:}-1$ resonance, i.e., two harmonic oscillators of $+1$ and -1 frequencies [3]. In general, a singular pinched torus is associated to an isolated singularity of the energy-momentum set [see Figs. 1(b) and 1(c) for an example].

The analysis of the geometry of singular tori and their importance in the dynamics of integrable Hamiltonians have been established only recently in the mathematical community. The fundamental problem was the generalization of action-angle variables to the whole phase space when the set of regular points (H, K) is not simply con-

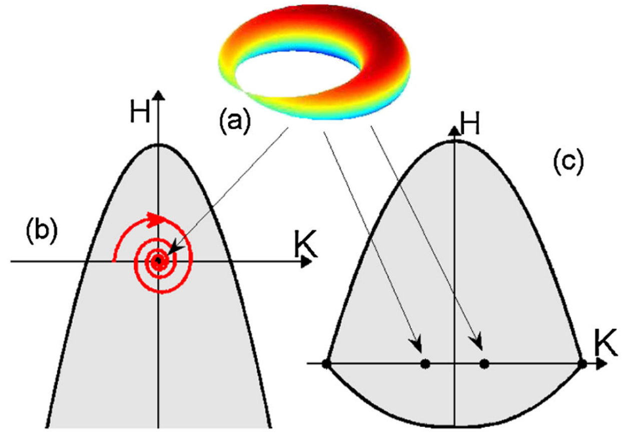

FIG. 1 (color online). Image of the energy-momentum map $\mathcal{F}$ (gray) for the Hamiltonian (2) (b) and the Hamiltonian (4) (c). The singular points are represented by solid lines and black dots. The dots indicate the positions of the image of the pinched tori (a). The red spiral path in (b) schematically represents the evolution $[\tilde{H}(t), \tilde{K}(t)]$ of the PDE system (1) during the space-time relaxation process [also see Figs. 2(c) and 2(d)].

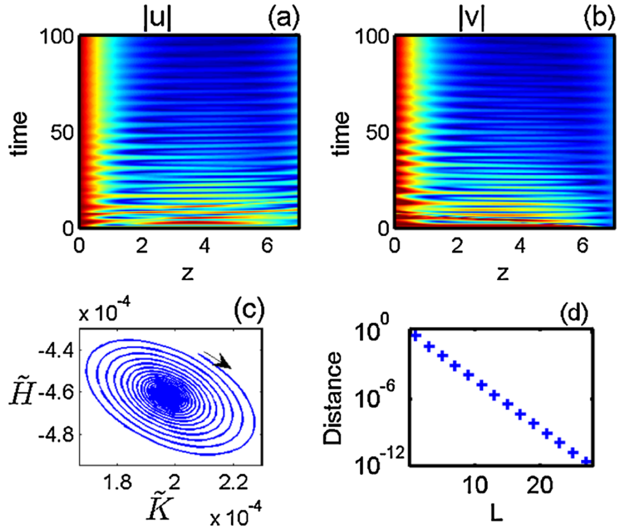

FIG. 2 (color online). (a)–(b) Numerical simulations of the PDE (1) showing the spatiotemporal evolution of the fields $|u|$ (a) and $|v|$ (b): after a transient the two waves relax to the stationary state associated to the singular pinched torus located at $H = K = 0$ [see Fig. 1(b)]. (c) Corresponding trajectory $[\tilde{H}(t), \tilde{K}(t)]$ in the energy-momentum set (H, K). The small arrow in (c) indicates the temporal evolution of the trajectory [$\kappa = 1$, $\gamma = 0.5$, $u_0 = 1$ (red), $v_L = 0.25$ (blue)]. (d) Evolution of the distance $\rho = \sqrt{H^2 + K^2}$ between the singular torus and the regular torus on which lies the stationary state for a given length L: the stationary state converges exponentially towards the singular torus as $L \to +\infty$.

nected. This precisely occurs when the image of $\mathcal{F}$ possesses an isolated singularity associated to a pinched torus. Duistermaat solved this problem in 1980 by introducing the concept of monodromy [2,4]. Nontrivial monodromy implies that after a loop in the regular values of the energy-momentum set, the action variables are changed. Different examples of monodromy in classical and quantum physics have been subsequently discovered [3,5], in particular, in the rovibrational spectrum of the CO_2 molecule [6], in the hydrogen atom in electric and magnetic fields, or in the problem of scattering by a central potential [7].

Our aim in this Letter is to show that singular tori play a major role in a novel class of physical systems. We deal here with the counterpropagating spatiotemporal dynamics of a nonlinear wave system which is ruled by partial differential equations (PDEs), i.e., a dynamical system with an infinite number of degrees of freedom. The numerical simulations reveal that, under rather general conditions, the spatiotemporal dynamics relaxes towards a stationary state. This property allows us to establish a relation between the original PDE and the corresponding integrable Hamiltonian dynamics ruled by the stationary ordinary differential equation. As a remarkable result, such a stationary state is shown to converge exponentially to the singular torus of the phase space of the associated ordinary differential equation.

The peculiar topological property inherent to the singular torus manifests itself by a robust character of the stationary solution of the PDE; i.e., the stationary solution preserves its global form regardless of the considered system size L. A consequence of this robust character is that the Hamiltonian system exhibits an attraction process: a wave injected from one side of the medium is attracted, independently of its initial state, towards a unique state determined by the properties of the singular torus. The space-time relaxation phenomenon and the associated process of attraction are exemplified by several key models of nonlinear wave interactions.

Let us begin to consider the evolution of a nonlinear wave in a periodic potential. This problem is encountered in a variety of physical disciplines such as optics, condensed matter physics, or Bose-Einstein condensates [10–12], in which the behavior of atoms mimics those of electrons in crystals or photons in optical gratings. Because of Bragg-reflections inherent to wave propagation in a periodic potential, these systems are generally characterized by a counterpropagating wave interaction [11,12]. More specifically, we analyze the one-dimensional model that rules nonlinear wave propagation around a forbidden frequency band gap

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} &= i\kappa v + i\gamma(|u|^2 + 2|v|^2)u, \\ \frac{\partial v}{\partial t} - \frac{\partial v}{\partial z} &= i\kappa u + i\gamma(|v|^2 + 2|u|^2)v, \end{aligned} \tag{1}$$

where κ and γ refer to the linear and nonlinear coefficients, u and v being the forward and backward complex wave amplitudes. In optics, Eq. (1) may be derived from Maxwell's equations, while in Bose-Einstein condensates they may be derived from the nonlinear Schrödinger equation with a periodic potential.

The counterpropagating nature of the interaction imposes the following boundary conditions: $u(0, t) = u_0$, $v(L, t) = v_L$. The numerical integration of (1) reveals that, after a complex transient, the two fields relax towards a stationary state associated to a singular torus (see Figs. 1 and 2). Indeed, to analyze the stationary solutions of Eqs. (1), we remark that the corresponding ordinary differential equation's system is Hamiltonian with respect to the canonically conjugate real coordinates (p_u, q_u, p_v, q_v) defined by $u = q_u + ip_u$ and $v = q_v - ip_v$,

$$H = \kappa(p_u p_v - q_u q_v) - \gamma|u|^2|v|^2 - \gamma(|u|^4 + |v|^4)/4. \tag{2}$$

The momentum $K = (|u|^2 - |v|^2)/2$ being a constant of motion, this system is Liouville integrable. We underline that the invariant K corresponds to a $1{:}-1$ resonance that originates in the counterpropagating nature of the wave interaction. The bifurcation diagram (H, K) of this system has been constructed by using the mathematical tools of Ref. [3] [see Fig. 1(b)]. Each point of the gray region in

Fig. 1(b) lifts to a regular torus in the phase space, whereas singular points are indicated by solid lines and dots lifted to singular tori. The numerical simulations show that the spatiotemporal dynamics is attracted towards a stationary state associated to a pinched torus, whose image is located at $H = K = 0$. This relaxation process is illustrated in Figs. 1(b) and 2(c), which show the temporal evolutions of $\tilde{H}(t) = \frac{1}{L}\int_0^L H(z,t)dz$ and $\tilde{K}(t) = \frac{1}{L}\int_0^L K(z,t)dz$. In a loose sense, $\tilde{H}$ and $\tilde{K}$ may be regarded as the instantaneous Hamiltonian and momentum averaged over the length L. The trajectories in Figs. 1(b) and 2(c) thus represent the spatiotemporal relaxation of the system towards the pinched torus ($H = K = 0$). Note that, for a finite length L, the relaxation occurs to a point in the neighborhood of the pinched torus $H \simeq K \simeq 0$ [see Fig. 2(c)]. Actually, the numerical simulations reveal that the stationary state converges exponentially towards the pinched torus in the limit $L \to +\infty$, as illustrated in Fig. 2(d). We conjecture that this exponential law originates in the logarithmic divergence of one of the periods of the torus as the distance between this torus and the pinched torus tends to zero.

This process of attraction towards a singular torus allows the interpretation of a recent experiment realized in optical fibers [13]. This effect was called polarization attraction due to its role in the dynamics of wave polarizations. Let us consider the counterpropagating four-wave interaction ruled by the following PDE:

$$\frac{\partial\vec{S}}{\partial t} + \frac{\partial\vec{S}}{\partial z} = [\vec{S}\times\mathcal{J}\vec{S} + 2\vec{S}\times\mathcal{J}\vec{J}],$$
$$\frac{\partial\vec{J}}{\partial t} - \frac{\partial\vec{J}}{\partial z} = [\vec{J}\times\mathcal{J}\vec{J} + 2\vec{J}\times\mathcal{J}\vec{S}]. \tag{3}$$

In the Poincaré-Stokes formalism, $\vec{S} = (S_x, S_y, S_z)$ and $\vec{J} = (J_x, J_y, J_z)$ respectively represent the polarizations of the forward and backward waves, while $\mathcal{J}$ denotes the diagonal matrix $\mathrm{diag}(-1, 0, -1)$ [14].

The phase space of the stationary ordinary differential equations (3) is the product of two spheres, $\mathbb{S}^2 \times \mathbb{S}^2$. The dynamics of the stationary state is governed by an integrable Hamiltonian

$$H = 2(S_xJ_x + S_zJ_z) - (S_y^2 + J_y^2)/2, \tag{4}$$

and the momentum $K = S_y + J_y$. On the basis of Ref. [8], we constructed the corresponding bifurcation diagram illustrated in Fig. 1(c). The space-time dynamics of the PDEs (3) is shown to be attracted towards one of the two singular pinched tori. As for the model (1), the convergence towards the singular torus follows an exponential law (data not shown). This exponential convergence plays an essential role for the experimental applications of the attraction process; in particular, it permitted the observation of polarization attraction within a very short nonlinear medium of length L [13].

Remarking that the pinched torus is a particular example of a singular torus encountered in integrable Hamiltonians [3], one may wonder whether other kinds of singular tori could be relevant to nonlinear wave systems. We illustrate this aspect by considering the example of the three-wave interaction, which is known to occur in any weakly nonlinear medium whose lowest order nonlinearity is quadratic in terms of the wave amplitudes. For this reason the three-wave interaction is encountered in such diverse fields as plasma physics, hydrodynamics, acoustics, and nonlinear optics [15]. In its counterpropagating configuration, the three-wave interaction model takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial z} = -vw, \quad \frac{\partial v}{\partial t} + \frac{\partial v}{\partial z} = uw^*, \quad \frac{\partial w}{\partial t} - \frac{\partial w}{\partial z} = uv^*, \tag{5}$$

where u and v (w) refer to the forward (backward) propagating waves. Introducing the canonically conjugated real coordinates $u = q_u + ip_u$, $v = q_v + ip_v$, and $w = q_w - ip_w$, the three degrees of freedom Hamiltonian associated to the stationary PDEs (5) reads

$$H = -q_vq_wp_u - p_vp_wp_u - q_vp_wq_u + p_vq_wq_u. \tag{6}$$

The constants of motion are $J = (|w|^2 - |u|^2)/2$ and $N = (|u|^2 + |w|^2 + 2|v|^2)$, which, respectively, correspond to the 1: $-$ 1 and 1:1:2 resonances. This Hamiltonian system exhibits a rather complex three-dimensional bifurcation diagram that can be constructed using techniques developed in [9]. We remark that this set is also encountered in the classical dynamics of the Fermi model of the CO_2 molecule [6]. It is characterized by a line of singularities for $J = H = 0$, whose preimage is a singular torus that is

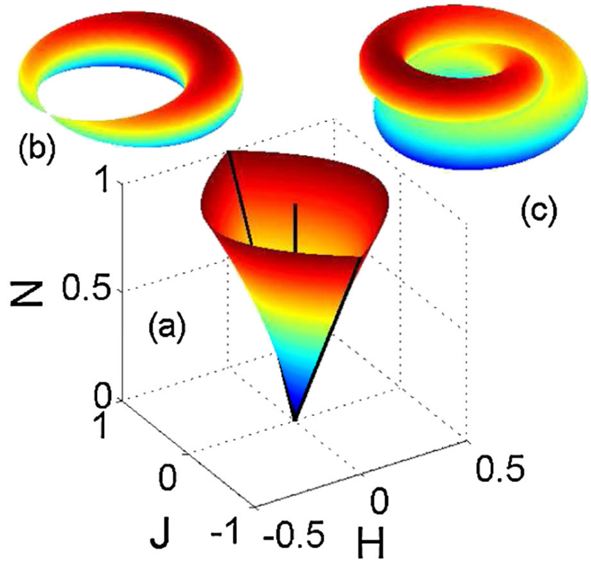

FIG. 3 (color online). (a) Bifurcation diagram of the energy-momentum map $\mathcal{F}:(q_u, p_u, q_v, p_v, q_w, p_w) \to (H, N, J)$. Each point of the singular line ($H = J = 0$) lifts in phase space to a singular torus which is both pinched (b) and curled (c).

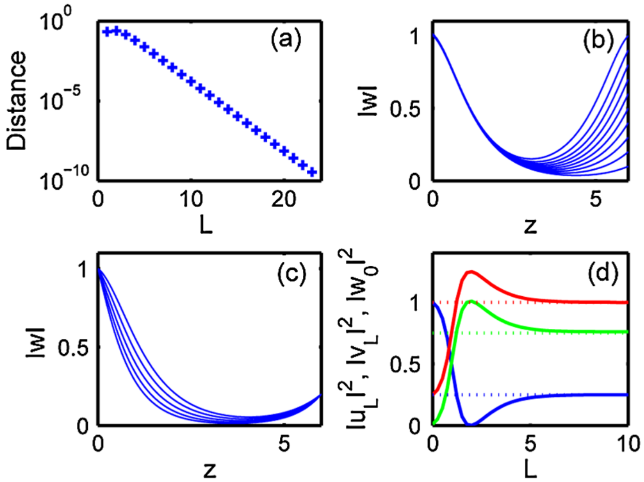

FIG. 4 (color online). Properties of the stationary solutions obtained by solving numerically the PDEs (5). (a) Exponential convergence of the stationary solution towards the pinched-curled torus as $L \to \infty$ (the distance between the regular and the singular torus is given by $\rho = \sqrt{J^2 + H^2}$). (b) The field w is attracted towards 1 regardless of its boundary condition w_L $[u_0 = 1,\ v_0 = 0.2]$. (c) Stationary states of w for different boundary conditions v_0 ranging from 0 to 1 $[u_0 = 1]$. (d) Dependence on L of u_L (blue), v_L (green), and w_0 (red), keeping fixed the boundary conditions ($u_0 = 1$, $v_0 = 0.1$, $w_L = 0.5$): the fields asymptotically reach the values corresponding to the singular torus (horizontal dashed lines).

both pinched and curled: in the subspace $N =$ const the torus is pinched, whereas it is curled in the subspace $J =$ const, as illustrated in Fig. 3. Note that a curled torus can be constructed by gluing together two cylinders along a line and identifying the extremities after a half-twist.

The numerical simulations of the PDEs (5) reveal that the space-time dynamics is attracted towards a stationary state, in a way akin to the relaxation process illustrated in Figs. 2(a) and 2(b). The stationary state lies on a regular torus, which is shown to converge exponentially towards the singular torus at ($J = 0$, $H = 0$) when the interaction length $L \to \infty$ [see Fig. 4(a)]. Accordingly, the stationary solution satisfies $|u(z)| = |w(z)|$, which means that the backward wave w is attracted towards the state $u_0 = u(z = 0)$, regardless of its initial condition at $w_L = w(z = L)$. This attraction process is clearly visible in the numerical simulations of the PDEs (5), as illustrated in Fig. 4(b). Note that the attraction process takes place irrespective of the boundary condition $v_0 = v(z = 0)$, as revealed by Fig. 4(c). We underline that, because of the singular nature of the torus, the global form of the stationary solutions reported in Figs. 4(b) and 4(c) is preserved for larger nonlinear interaction lengths L. This remarkable robustness is illustrated in Fig. 4(d), in which we report the output values of the fields $[u_L = u(z = L),\ v_L = v(z = L),\ w_0 = w(z = 0)]$ for increasing lengths (L) of interaction, keeping fixed the boundary conditions ($u_0 = 1$, $v_0 = 0.1$, $w_L = 0.5$). For large L, the fields asymptotically reach the values predicted by the singular torus, i.e., $u_L^2 = w_L^2$, $w_0^2 = N - u_0^2 - 2v_0^2$, $v_L^2 = N/2 - w_L^2$ with $N = 2(u_0^2 + v_0^2)$.

In summary, through the analysis of three fundamental models of spatiotemporal wave interactions, we have shown that the peculiar topological properties of singular tori play a previously unrecognized fundamental role in the dynamics of a wave system. Within these models equations, we also showed that the asymptotic dynamics of a system with an infinite number of degrees of freedom is captured by a low-dimensional dynamical system. Given the ubiquitous character of the considered model equations, the phenomenon of attraction reported here may be transposed to a large variety of physical systems.

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