Entangling and Disentangling Power of Unitary Transformations Are Not Equal

Noah Linden,[1,*] John A. Smolin,[2,†] and Andreas Winter[1,‡]

[1]Department of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom

[2]IBM T. J. Watson Research Center, Yorktown Heights, New York 10598, USA

(Received 14 August 2008; revised manuscript received 25 May 2009; published 14 July 2009)

We consider two capacity quantities associated with bipartite unitary gates: the entangling and the disentangling power. Here, we prove that these capacities are different in general by constructing an explicit example of a qubit-qutrit unitary whose entangling power is maximal (2 ebits), but whose disentangling power is strictly less. A corollary is that there can be no unique ordering for unitary gates in terms of their ability to perform nonlocal tasks. Finally, we show that in large dimensions, almost all bipartite unitaries have entangling and disentangling capacities close to maximal.

DOI: 10.1103/PhysRevLett.103.030501 PACS numbers: 03.67.Mn, 03.65.Ta, 03.65.Ud

Given two interacting quantum systems, a fundamental issue is to quantify the strength of this interaction. The question is straightforward if one is comparing two situations in which the particles and physical nature of the interaction are the same; it is then easy to say one interaction is stronger than the other. However, one would like characterizations of interaction strength that go well beyond this. Particularly valuable are techniques that can compare interactions of quite different types of system or particles and different physical manifestations of the interaction. As well as the fundamental nature of this question, the issue is clearly of interest to experimentalists who try to create systems in interaction. Robust and general characterizations of interaction strength provide an intellectual background to compare different physical systems. Quantum information theory has provided quite new insights into this fundamental question. It is now understood that one can talk about an interaction between systems as an abstract notion, not needing to say how this interaction arose. In particular, there has been considerable progress in quantifying the strength of Hamiltonian and unitary interactions [1–11]. The starting point was the theory of entanglement of quantum *states*; entanglement quantifies how much nonclassical correlation the state embodies. For example, for pure quantum states, we understand that the entropy of entanglement is essentially the unique measure of the entanglement in a state, and the *ebit*, the amount of entanglement in a singlet state, is the natural unit of entanglement. The amount of entanglement in a quantum state is a notion that makes no reference to the physical nature of the systems involved. In due course, there was the realization that the strength of an interaction could be quantified by how well it can create entanglement in states [1–4]. Thus, for example, the *entangling capacity* of a unitary interaction is defined as the maximum amount of entanglement increase it can produce when acting on a quantum state. For example, the unitary SWAP operation between two qubits can create two ebits, and the CNOT operation can create one ebit. A particularly attractive aspect of this approach is that the entangling capacity can be defined for bipartite systems of any size, and thus it is meaningful to compare systems whose state spaces are different sizes. While this is very attractive in that it quantifies the strength of an interaction abstractly, it was also understood that many other natural measures arise. For example, one can ask how much entanglement is needed to *create* a quantum operation [for the SWAP and CNOT operations, this is two and one ebit, respectively [2–4]]. But it was also realized that quantum operations can do other things: they can also perform classical communication, for example. One can also ask how good interactions are at *disentangling*.

The systems that were most well studied are systems of two qubits. In this case, the situation seems particularly simple. While there are many questions one could ask about a unitary interaction, in fact, it seems to be the case, roughly speaking, that when comparing two interactions, when an interaction U is stronger than an interaction V in one sense, it also was typically stronger in any other sense as well [when these strengths could be computed]. A particular case in point is the power to create or destroy entanglement; here, one can prove that these two quantities are the same for unitary interactions of two qubits. This leads to the very attractive conjecture that in fact there is a universal measure of the strength of quantum interactions. The key upshot of the results of this Letter is that this conjecture is not true. Our main technical result is that the entangling power and disentangling power of unitary interactions are unequal in general. This leads to a simple corollary that there are unitaries, U_1 and U_2, for which U_1 is stronger than U_2 according to one natural measure of their interaction strength but for which the ordering is the other way around, according to another measure.

Formally, we consider a unitary transformation U acting on a bipartite system shared by two observers Alice and Bob. Alice has a system Hilbert space $\mathcal{H}_A$ and Bob a system Hilbert space $\mathcal{H}_B$. The unitary $U = U_{AB}$ acts on $\mathcal{H}_A \otimes \mathcal{H}_B$. Alice (resp. Bob) also has an ancilla with Hilbert space $\mathcal{H}_a$ (resp. $\mathcal{H}_b$). We consider an initial state $|\Psi^{\rm in}\rangle$ on the full Hilbert space, then act with $I_a \otimes U_{AB} \otimes I_b$

to produce a final state

$$|\Psi^{\rm out}\rangle = I_a \otimes U_{AB} \otimes I_b|\Psi^{\rm in}\rangle. \quad (1)$$

Let $E(\Psi^{\rm in})$ be the entanglement of $|\Psi^{\rm in}\rangle$, measured by the entropy of its reduced state on the space $\mathcal{H}_A \otimes \mathcal{H}_a$ [12]. Then, the entangling power of U, which we denote $E^{\uparrow}(U)$, is defined to be the maximum possible increase in the entanglement as the input state varies:

$$E^{\uparrow}(U) = \sup_{|\Psi^{\rm in}\rangle}[E(\Psi^{\rm out}) - E(\Psi^{\rm in})]. \quad (2)$$

Similarly, we define the disentangling power

$$E^{\downarrow}(U) = \sup_{|\Psi^{\rm in}\rangle}[E(\Psi^{\rm in}) - E(\Psi^{\rm out})]. \quad (3)$$

Clearly, $E^{\downarrow}(U) = E^{\uparrow}(U^{\dagger})$. Note that by the results of [9,10], $E^{\uparrow}(U)$ is equal to the asymptotic (many copies of U) capacity of U to generate entanglement and that optimization over pure states is sufficient.

In this Letter, we prove that in general $E^{\uparrow}(U)$ and $E^{\downarrow}(U)$ are not equal. We show this by constructing an explicit example in 2×3 dimensions. Recalling that for 2×2-unitaries $E^{\uparrow}(U) = E^{\downarrow}(U) = E^{\uparrow}(U^{\dagger})$ [8], we note our example occurs in the smallest possible dimension.

We have not said anything up to this point about the relative dimensions of the system and ancilla Hilbert spaces. It is known that for typical unitaries U, it is essential to have ancillas in order to generate the maximum possible entanglement using U. A well-known extreme case is the two-qubit SWAP: it generates no entanglement increase if Alice and Bob each only have the qubit on which the SWAP acts, but it generates two ebits, the maximum increase for any unitary acting on two qubits, if Alice and Bob each have an additional qubit ancilla. For an arbitrary U, it is not known what size the ancillas need to be to reach the maximum possible entanglement increase (or decrease) for that unitary, or if indeed a maximizer exists in finite dimension. Until now, this has been a major stumbling block in the calculation of the nonlocal capacities of interactions [9,10].

Main result.—Our proof that the entangling and disentangling power are unequal proceeds in two steps. First, we show that if a unitary transformation has the largest possible entangling power for a unitary of that dimension, then the local ancillas need only be as large as the local system Hilbert spaces. Then, we exhibit an explicit 2×3 unitary whose entangling power is maximal (2 ebits) but its disentangling power is strictly less than 2 ebits.

Lemma 1 Let U be a unitary acting on $\mathbb{C}^A \otimes \mathbb{C}^B$ $(A \le B)$.—If U is maximally entangling [i.e., $E^{\uparrow}(U) = 2\log A$ [13]], then in Eqs. (1) and (2), one may restrict to ancillas of dimension $a = A$ and $b = B$; in particular, the supremum is a maximum, achieved using an input state of the product form $|\Psi^{\rm in}\rangle_{aABb} = |\Phi\rangle_{aA} \otimes |\Psi\rangle_{Bb}$, *with* $|\Phi\rangle_{aA} = \frac{1}{\sqrt{A}}\sum_{j=1}^{A}|j\rangle_a|j\rangle_A$ a maximally entangled state on $a \times A$ and some $|\Psi\rangle_{Bb}$ on $B \times b$.

Proof.—First, assume that for some ancillas of size a and b, respectively, there is actually a maximizer $|\Psi^{\rm in}\rangle$. In the supplementary information [14], we give a proof that avoids this unwarranted assumption. Generally, subadditivity of entropy [15] implies the entanglement of the final state $E(\Psi^{\rm out})$ satisfies

$$E(\Psi^{\rm out}) = S(\rho^{\rm out}_{Aa}) \le S(\rho_a) + \log A. \quad (4)$$

Also, the triangle inequality [15] implies that

$$E(\Psi^{\rm in}) = S(\rho^{\rm in}_{Aa}) \ge S(\rho_a) - \log A. \quad (5)$$

[Notice that since the unitary U does not act on the ancilla Hilbert space, $S(\rho_a)$ is the same before and after the action of U.] Thus, $E(\Psi^{\rm out}) - E(\Psi^{\rm in}) \le 2\log A$, but since we assumed that $E(\Psi^{\rm out}) - E(\Psi^{\rm in}) = 2\log A$, we must have equality in Eqs. (4) and (5).

Now we can calculate, using the above and the purity of the state of four parties,

$$S(\rho^{\rm in}_{ABb}) = S(\rho_a) = S(\rho^{\rm in}_{Aa}) + \log A = S(\rho^{\rm in}_{Bb}) + \log A.$$

Thus, we must have $\rho^{\rm in}_{ABb} = \frac{1}{A}I_A \otimes \rho^{\rm in}_{Bb}$, and we may purify the state $\rho^{\rm in}_{ABb}$ by writing Alice's ancilla Hilbert space in the form $\mathcal{H}_{a_1} \otimes \mathcal{H}_{a_2}$ so that the full state is

$$|\Psi^{\rm in}\rangle_{aABb} = |\Psi^{\rm in}_1\rangle_{a_1A} \otimes |\Psi^{\rm in}_2\rangle_{a_2Bb}. \quad (6)$$

We may take a_1 to have dimension A and $|\Psi^{\rm in}_1\rangle_{Aa_1}$ is maximally entangled, and hence $\rho_{a_1} = \frac{1}{A}I_{a_1}$.

We now consider the state after the action of U. Equation (4) with equality means that $\rho^{\rm out}_{Aa} = \frac{1}{A}I_A \otimes \rho_a$, so that

$$\rho^{\rm out}_{Aa_1a_2} = \frac{1}{A}I_A \otimes \rho_a = \frac{1}{A}I_A \otimes \frac{1}{A}I_{a_1} \otimes \rho_{a_2}. \quad (7)$$

Hence, using Eqs. (6) and (7), we have

$$E(\Psi^{\rm in}) = S(\rho_{a_2}) \quad \text{and} \quad E(\Psi^{\rm out}) = S(\rho_{a_2}) + 2\log A.$$

We may now see that there is a different initial state yielding the same entanglement increase. We take exactly the state (6) but now consider the situation in which the ancilla particle a_2 is transferred to Bob—let us relabel $\tilde{a} = a_1$ and $\tilde{b} = ba_2$. Thus, consider the initial state

$$|\tilde{\Psi}^{\rm in}\rangle_{\tilde{a}AB\tilde{b}} = |\Psi^{\rm in}_1\rangle_{\tilde{a}A} \otimes |\Psi^{\rm in}_2\rangle_{B\tilde{b}}. \quad (8)$$

This state has $E(\tilde{\Psi}^{\rm in}) = 0$ and $E(\tilde{\Psi}^{\rm out}) = 2\log A$.

The above description assumes that the supremum in (2) is a maximum. The result may be proved without this assumption; the proof is rather involved and may be found in the supplementary information [14]. □

We thus conclude that if a unitary creates maximal entanglement, it can do so by acting on a product pure state between Alice and Bob. Furthermore, Alice's state may be taken to be maximally entangled between the system and ancilla. If $A < B$ may only conclude that Bob's initial state can be pure with an ancilla of dimension B.

Of course, if $A = B$, we may run the argument again, with the roles of Alice and Bob interchanged, to show that the initial state may be taken to be a product state between Alice and Bob, with both maximally entangled with their local ancillas. From this, it is not hard to show that, still assuming that U is maximally entangling, it is also maximally disentangling. In other words, for $A = B$,

$$E^{\uparrow}(U) = 2\log A \Leftrightarrow E^{\downarrow}(U) = 2\log A.$$

We now exhibit explicitly a 2×3 unitary which can entangle, but not disentangle, maximally

$$U_{2\times 3} = -i|w_{00}\rangle\langle 00| + |w_{01}\rangle\langle 01| + |w_{02}\rangle\langle 02| + |w_{10}\rangle\langle 10| + |w_{11}\rangle\langle 11| - i|w_{12}\rangle\langle 12|, \tag{9}$$

with (for $j = 0, 1, 2$)

$$|w_{0j}\rangle = \frac{1}{\sqrt{3}}(|\alpha\rangle|0\rangle + \omega^j|\beta\rangle|1\rangle + \omega^{2j}|\gamma\rangle|2\rangle),$$

$$|w_{1j}\rangle = \frac{1}{\sqrt{3}}(|\alpha^\perp\rangle|0\rangle + \omega^j|\beta^\perp\rangle|1\rangle + \omega^{2j}|\gamma^\perp\rangle|2\rangle).$$

Here, $\omega = e^{2\pi i/3}$ and $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ are the "trine" states

$$|\alpha\rangle = |0\rangle, \qquad |\beta\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle,$$

$$|\gamma\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$$

with $|\alpha^\perp\rangle$, $|\beta^\perp\rangle$, $|\gamma^\perp\rangle$ their orthogonal complements, respectively, chosen with real coefficients.

$U_{2\times 3}$ can create two ebits. Consider its action on

$$|\Phi_1^{\text{in}}\rangle = \frac{1}{2}(|0\rangle_a|0\rangle_A + |1\rangle_a|1\rangle_A) \otimes (|0\rangle_B|0\rangle_b + |2\rangle_B|2\rangle_b).$$

The subscript A denotes Alice's system and a her ancilla, similarly for Bob. Clearly, the initial state $|\Phi_1^{\text{in}}\rangle$ has zero entanglement between Aa and Bb. It is not difficult to check that the final state $|\Phi_1^{\text{out}}\rangle = U_{2\times 3}|\Phi_1^{\text{in}}\rangle$ has entanglement of two ebits. Thus, $U_{2\times 3}$ has the maximum possible entangling power for any unitary on $\mathbb{C}^2 \otimes \mathbb{C}^3$.

We now show that the disentangling power of $U_{2\times 3}$ is strictly less than 2 ebits. It will be convenient to analyze the entangling power of $U_{2\times 3}^\dagger$, the inverse of $U_{2\times 3}$. If the entangling power of $U_{2\times 3}^\dagger$ were 2 ebits, then it could be achieved, following Lemma 1, starting with a product state: $|\eta_1\rangle_{aA} \otimes |\eta_2\rangle_{Bb}$, where $|\eta_1\rangle_{aA}$ is a maximally entangled state of two qubits, $|\eta_2\rangle_{Bb}$ an arbitrary pure state of two qutrits. The proof will consist in showing that one cannot achieve 2 ebits starting with a state of this form.

Thus, the most general input state we need to consider is

$$|\Phi_2^{\text{in}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_A + |1\rangle_a|1\rangle_A) \otimes (|0\rangle_B|\tau_0\rangle_b + |1\rangle_B|\tau_1\rangle_b + |2\rangle_B|\tau_2\rangle_b). \tag{10}$$

Normalization of $|\Phi_2^{\text{in}}\rangle$ means that $\langle\tau_0|\tau_0\rangle_b + \langle\tau_1|\tau_1\rangle_b + \langle\tau_2|\tau_2\rangle_b = 1$. Clearly, $|\Phi_2^{\text{in}}\rangle$ has no entanglement between Aa and Bb.

To compute the output state, we begin by rewriting the inverse $U_{2\times 3}^\dagger$ as

$$\begin{aligned} U_{2\times 3}^\dagger = {} & |0\rangle_A|v_0\rangle_B\langle 0|_A\langle 0|_B + \Big[-\frac{1}{2}|0\rangle_A|v_1\rangle_B - \frac{\sqrt{3}}{2}|1\rangle_A|v_1'\rangle_B\Big]\langle 0|_A\langle 1|_B + \Big[-\frac{1}{2}|0\rangle_A|v_2\rangle_B \\ & + \frac{\sqrt{3}}{2}|1\rangle_A|v_2'\rangle_B\Big]\langle 0|_A\langle 2|_B + |1\rangle_A|v_0'\rangle_B\langle 1|_A\langle 0|_B \\ & + \Big[\frac{\sqrt{3}}{2}|0\rangle_A|v_1\rangle_B - \frac{1}{2}|1\rangle_A|v_1'\rangle_B\Big]\langle 1|_A\langle 1|_B \\ & + \Big[-\frac{\sqrt{3}}{2}|0\rangle_A|v_2\rangle_B - \frac{1}{2}|1\rangle_A|v_2'\rangle_B\Big]\langle 1|_A\langle 2|_B, \end{aligned}$$

where (for $j = 0, 1, 2$)

$$|v_j\rangle = \frac{1}{\sqrt{3}}(i|0\rangle + \omega^{-j}|1\rangle + \omega^{-2j}|2\rangle),$$

$$|v_j'\rangle = \frac{1}{\sqrt{3}}(|0\rangle + \omega^{-j}|1\rangle + i\omega^{-2j}|2\rangle).$$

Thus, the result of $U_{2\times 3}^\dagger$ acting on (10) is

$$\begin{aligned} |\Phi_2^{\text{out}}\rangle &= U_{2\times 3}^\dagger|\Phi_2^{\text{in}}\rangle \\ &= \frac{1}{2}[|0\rangle_a|0\rangle_A|\Phi_{00}\rangle_{Bb} + |0\rangle_a|1\rangle_A|\Phi_{01}\rangle_{Bb} \\ &\quad + |1\rangle_a|0\rangle_A|\Phi_{10}\rangle_{Bb} + |1\rangle_a|1\rangle_A|\Phi_{11}\rangle_{Bb}], \end{aligned}$$

where now

$$|\Phi_{00}\rangle = \sqrt{2}\Big[|v_0\rangle_B|\tau_0\rangle_b - \frac{1}{2}|v_1\rangle_B|\tau_1\rangle_b - \frac{1}{2}|v_2\rangle_B|\tau_2\rangle_b\Big],$$

$$|\Phi_{01}\rangle = \sqrt{2}\Big[-\frac{\sqrt{3}}{2}|v_1'\rangle_B|\tau_1\rangle_b + \frac{\sqrt{3}}{2}|v_2'\rangle_B|\tau_2\rangle_b\Big],$$

$$|\Phi_{10}\rangle = \sqrt{2}\Big[\frac{\sqrt{3}}{2}|v_1\rangle_B|\tau_1\rangle_b - \frac{\sqrt{3}}{2}|v_2\rangle_B|\tau_2\rangle_b\Big],$$

$$|\Phi_{11}\rangle = \sqrt{2}\Big[|v_0'\rangle_B|\tau_0\rangle_b - \frac{1}{2}|v_1'\rangle_B|\tau_1\rangle_b - \frac{1}{2}|v_2'\rangle_B|\tau_2\rangle_b\Big].$$

Now, in order for $|\Phi_2^{\text{out}}\rangle$ to be maximally entangled, we require that the four states $|\Phi_{00}\rangle$, $|\Phi_{01}\rangle$, $|\Phi_{10}\rangle$, and $|\Phi_{11}\rangle$ form an orthonormal basis. This puts constraints on the $|\tau_j\rangle$, which, as we shall see, leads to a contradiction.

Bearing in mind the normalization of the $|\tau_j\rangle$, the four equations expressing the condition that the vectors $|\Phi_{ij}\rangle$ be normalized are all the same, namely: $\langle\tau_1|\tau_1\rangle + \langle\tau_2|\tau_2\rangle = \frac{2}{3}$, or equivalently, $\langle\tau_0|\tau_0\rangle = \frac{1}{3}$. The requirement that $\langle\Phi_{00}|\Phi_{10}\rangle = 0$ thus leads to

$$\langle\tau_0|\tau_0\rangle = \langle\tau_1|\tau_1\rangle = \langle\tau_2|\tau_2\rangle = \frac{1}{3}. \tag{11}$$

The requirement that $\langle\Phi_{01}|\Phi_{10}\rangle = 0$ yields

$$-\langle\tau_1|\tau_1\rangle - \langle\tau_2|\tau_2\rangle + (1-\sqrt{3})\omega^2\langle\tau_1|\tau_2\rangle + (1+\sqrt{3})\omega\langle\tau_2|\tau_1\rangle = 0.$$

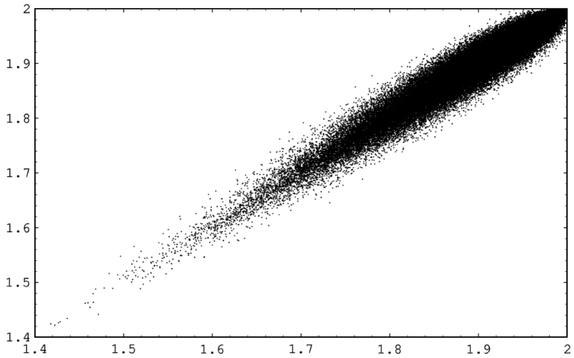

FIG. 1. Entangling vs disentangling power for 170 000 randomly chosen 2×3 unitaries. The local ancilla dimensions are 2 and 3, respectively.

This has the unique solution $\langle \tau_1 | \tau_2 \rangle = \frac{\omega}{3}$, and with the Cauchy-Schwarz inequality and Eq. (11), this means that

$$|\tau_2\rangle = \omega|\tau_1\rangle. \qquad (12)$$

The requirement that $\langle \Phi_{00} | \Phi_{01} \rangle = 0$ gives

$$-\omega^2(1+\sqrt{3})\langle \tau_0|\tau_1\rangle + \omega(1-\sqrt{3})\langle \tau_0|\tau_2\rangle + \frac{1}{2}\langle \tau_1|\tau_1\rangle - \frac{\omega^2}{2}(1+\sqrt{3})\langle \tau_1|\tau_2\rangle + \frac{\omega}{2}(1-\sqrt{3})\langle \tau_2|\tau_1\rangle - \frac{1}{2}\langle \tau_2|\tau_2\rangle = 0.$$

Using Eqs. (11) and (12), this implies that

$$\langle \tau_0|\tau_1\rangle = -\frac{\omega}{6} \quad \text{and} \quad \langle \tau_0|\tau_2\rangle = -\frac{\omega^2}{6}. \qquad (13)$$

But now, inserting Eqs. (11)–(13), we get $\langle \Phi_{00}|\Phi_{11}\rangle = \frac{2}{3} \neq 0$. Thus, there is no orthonormal choice of $|\tau_0\rangle$, $|\tau_1\rangle$, $|\tau_2\rangle$ for $|\Phi_{ij}\rangle$. This is the desired contradiction, and we conclude that $E^{\downarrow}(U) < 2 = E^{\uparrow}(U)$.

Conclusion.—We have found an example of bipartite unitary of smallest possible dimension such that its entangling and its disentangling power are different. This is a striking result as it shows that there can be no unique ordering of unitary gates with respect to their various capacities. For consider $U_1 = U_{2\times3}$ and $U_2 = U_{2\times3}^{\dagger}$: U_1 has greater entangling capacity than U_2, but U_1 has smaller disentangling capacity than U_2.

We have done numerical work, which, for $U_{2\times3}$, indicates that $2 - E^{\downarrow}(U_{2\times3}) \approx 0.06$. Furthermore, we tried to find the maximum difference $E^{\uparrow}(U) - E^{\downarrow}(U)$ over all 2×3 gates U, which seems to be ≈ 0.13, and in general for a random unitary, the entangling and the disentangling power are not much different. [See Fig. 1.]

We can explain this partly by the *concentration of measure phenomenon* [16] in large dimensions (which usually however kicks in for relatively small dimensions). Using arguments similar to those in [17], it may be shown that a random U (for large $A \leq B$ or for B much larger than A) has entangling and disentangling power close to the maximum of $2 \log A$ (and therefore the difference between these capacities is also likely to be small). We note that this does not preclude the possibility that a particular unitary could have very different entangling and disentangling power. Indeed, independent work by Harrow and Shor [18] shows that for large local dimensions d, it is possible to construct a unitary for which $E^{\uparrow}(U) - E^{\downarrow}(U) \sim \log d$.

N. L. and A. W. thank the EU for support through the European Commission project RESQ (Contract No. IST-2001-37559); N. L., A. W., and J. A. S. thank the UK EPSRC for support through the IRC-QIP. J. A. S. thanks the NSA and ARDA for support through Contract No. DAAD19-01-C-0056. We also would like to thank A. Harrow and P. Shor for conversations about their work.

*n.linden@bristol.ac.uk
†smolin@watson.ibm.com
‡a.j.winter@bris.ac.uk

[1] W. Dür, G. Vidal, J. I. Cirac, N. Linden, and S. Popescu, Phys. Rev. Lett. **87**, 137901 (2001).
[2] D. Collins, N. Linden, and S. Popescu, Phys. Rev. A **64**, 032302 (2001).
[3] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Phys. Rev. A **62**, 052317 (2000).
[4] A. Chefles, C. R. Gilson, and S. M. Barnett, Phys. Lett. A **273**, 10 (2000).
[5] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301 (2000).
[6] B. Kraus and J. I. Cirac, Phys. Rev. A **63**, 062309 (2001).
[7] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).
[8] D. W. Berry and B. C. Sanders, Phys. Rev. A **67**, 040302 (2003); Phys. Rev. A **68**, 032312 (2003).
[9] C. H. Bennett, A. W. Harrow, D. W. Leung, and J. A. Smolin, IEEE Trans. Inf. Theory **49**, 1895 (2003).
[10] M. S. Leifer, L. Henderson, and N. Linden, Phys. Rev. A **67**, 012306 (2003).
[11] M. A. Nielsen *et al.*, Phys. Rev. A **67**, 052301 (2003).
[12] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. W. Schumacher, Phys. Rev. A **53**, 2046 (1996).
[13] An easy way to see that $2 \log A$ is the maximum entanglement is to note that the gate can be simulated using this much entanglement: Alice teleports her system to Bob, who applies the gate and teleports her system back.
[14] See EPAPS Document No. E-PRLTAO-103-076930 for supplementary information. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.
[15] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978).
[16] M. Ledoux, The Concentration of Measure Phenomenon, AMS Mathematical Surveys and Monographs (2001), Vol. 89.
[17] P. Hayden, D. W. Leung, and A. Winter, Commun. Math. Phys. **265**, 95 (2006).
[18] A. W. Harrow and P. W. Shor, arXiv:quant-ph/0511219.