Exponentially Fragile $\mathcal{PT}$ Symmetry in Lattices with Localized Eigenmodes

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We study the effect of localized modes in lattices of size N with parity-time ($\mathcal{PT}$) symmetry. Such modes are arranged in pairs of quasidegenerate levels with splitting $\delta \sim \exp^{-N/\xi}$ where ξ is their localization length. The level "evolution" with respect to the $\mathcal{PT}$ breaking parameter γ shows a cascade of bifurcations during which a pair of real levels becomes complex. The spontaneous $\mathcal{PT}$ symmetry breaking occurs at $\gamma_{\mathcal{PT}} \sim \min\{\delta\}$, thus resulting in an exponentially narrow exact $\mathcal{PT}$ phase. As N/ξ decreases, it becomes more robust with $\gamma_{\mathcal{PT}} \sim 1/N^2$ and the distribution $\mathcal{P}(\gamma_{\mathcal{PT}})$ changes from log-normal to semi-Gaussian. Our theory can be tested in the frame of optical lattices.

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Introduction.—Parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetries, as well as their breaking, belong to the most basic notions in physics. Recently there has been much interest in systems which do not obey $\mathcal{P}$ and $\mathcal{T}$ symmetries separately but do exhibit a combined $\mathcal{PT}$ symmetry. Examples of such $\mathcal{PT}$-symmetric systems range from quantum field theories to solid-state and classical optics [1–9]. A $\mathcal{PT}$-symmetric system can be realized in optics, by creating a medium with alternating regions of gain and loss, such that the (complex) refraction index satisfies the condition $n^*(-x) = n(x)$ [6–9]. This condition implies that creation and absorption of photons occur in a balanced manner, so that the net loss or gain is zero. Such synthetic $\mathcal{PT}$ metamaterials show unique characteristics such as "double refraction" and nonreciprocal diffraction patterns, which may allow their use as a new generation of unidirectional optical couplers or left-right sensors of propagating light [6]. In the paraxial approximation the classical wave equations reduce to a Schrödinger equation with a fictitious time, related to the propagation distance, and with the refraction index playing the role of the potential. We use below the terminology of the Schrödinger equation, while keeping in mind applications to optical systems.

A $\mathcal{PT}$-symmetric system can be described by a phenomenological "Hamiltonian" $\mathcal{H}$. Such Hamiltonians may have a unitary time evolution and a real energy spectrum, although in general are nonhermitian. Furthermore, as some parameter of $\mathcal{H}$ changes, a spontaneous $\mathcal{PT}$ symmetry breaking occurs, at which point the eigenfunctions of $\mathcal{H}$ cease to be eigenfunctions of the $\mathcal{PT}$ operator, despite the fact that $\mathcal{H}$ and the $\mathcal{PT}$ operator commute [1]. This happens because the $\mathcal{PT}$ operator is not linear, and thus the eigenstates of $\mathcal{H}$ may or may not be eigenstates of $\mathcal{PT}$. As a consequence, in the *broken $\mathcal{PT}$-symmetry phase* the spectrum becomes partially or completely complex. The other limiting case where both $\mathcal{H}$ and $\mathcal{PT}$ share the same set of eigenvectors, corresponds to the so-called *exact $\mathcal{PT}$-symmetric phase* and the spectrum is real.

In this Letter we investigate the spontaneous $\mathcal{PT}$-symmetry breaking scenario in a wide class of systems supporting localized states. Such states are ubiquitous in macroscopic systems. They can reside on impurities or at the edges of an otherwise perfect lattice of finite size. Therefore, in order to understand the $\mathcal{PT}$-symmetric phase of a macroscopic system, it is imperative to consider localized states. At the same time, we note that even 50 years after the seminal work of Anderson [10], localization continues to be a thriving area of research, not only for solid-state physics, but also to other fields including ultracold atoms, acoustics, microwaves and classical optics. We therefore expect that our study linking the newly developed area of $\mathcal{PT}$ materials with the field of localization will contribute to understanding fundamental aspects of modern physics.

Localization is particularly pronounced in one-dimensional (1D) systems and has been studied extensively in the past [11]. We show that in the case of $\mathcal{PT}$-symmetric lattices of size N which can support localized modes due to disorder or impurities or even due to boundaries (surface states), the mechanism that triggers the transition from real to complex spectrum is level crossing between a pair of modes having the smallest energy spacing. Because of the $\mathcal{P}$ symmetry, this pair of states has a double hump shape and the energy splitting between them is $\delta_1 \sim \exp(-l_0/\xi)$ where ξ is the localization length and l_0 is the distance between the two humps (for disordered lattices $l_0 \sim N$). We find that the value of the $\mathcal{PT}$ symmetry breaking parameter γ at the transition point is $\gamma_{\mathcal{PT}} \sim \delta_1$, thus indicating that *the exact $\mathcal{PT}$-symmetric phase is exponentially small* in the limit $l_0/\xi \gg 1$. In contrast, for $l_0/\xi \ll 1$, we find that the smaller level spacing scales as $\Delta_{\min} \sim 1/N^2$. This is also reflected in the distribution $\mathcal{P}(\gamma_{\mathcal{PT}})$ which changes from a log-normal towards a semi-Gaussian as N/ξ decreases.

Two $\mathcal{PT}$-symmetric impurities.—It is instructive to start with the simple example of a pair of $\mathcal{PT}$-symmetric impurities implanted into an otherwise perfect infinite lattice. The system is described by the equation

$$-\psi_{n+1} - \psi_{n-1} = (E - \varepsilon_n)\psi_n, \tag{1}$$

where ψ_n is the eigenfunction amplitude at site n, $\varepsilon_n = 0$ for $n \neq \pm l$, and $\varepsilon_{\pm l} = -\beta \pm i\gamma$, with β and γ being real and positive. We are looking for the bound states:

$$\psi_n = \begin{cases} Ae^{kn}, & n \leq -l \\ Be^{kn} + Ce^{-kn}, & -l \leq n \leq l \\ De^{-kn}, & n \geq l \end{cases} \tag{2}$$

with $\mathrm{Re}[k] > 0$ and $E = -2\cosh k$. Matching the wave function at the sites $n = \pm l$, one obtains four equations for the amplitudes A, B, C, D. Equating the determinant to zero yields the transcendental equation for k:

$$\sinh k = \frac{\beta}{2} \pm \frac{1}{2}[-\gamma^2 + (\beta^2 + \gamma^2)e^{-4kl}]^{1/2}. \tag{3}$$

For $\gamma = 0$ and $\beta l \gg 1$, one finds two bound states with energies $E_\pm = E_0 \pm \frac{1}{2}\delta_1$, where $E_0 = -\sqrt{4 + \beta^2}$ is the energy of a localized state on a single isolated impurity, and $\delta_1 = \delta(l) = (2\beta^2/|E_0|)e^{-\beta l}$ is the exponentially small energy splitting term for the two-impurity problem. The point we want to emphasize is that for $\beta l \gg 1$ already an exponentially small γ leads to complex values of k and E, thus, breaking the $\mathcal{PT}$ symmetry. The mechanism for this breaking is level crossing: as follows from Eq. (3), when γ reaches the value $\gamma_{\mathcal{PT}} \approx \beta e^{-\beta l}$, the two (real) eigenvalues become degenerate. For $\gamma > \gamma_{\mathcal{PT}}$ they branch out into the complex plane, displaying near the branch point the characteristic behavior $\mathrm{Im}[E_\pm] \propto \pm\sqrt{\gamma^2 - \gamma_{\mathcal{PT}}^2}$. This square-root singularity seems to be a generic feature of the $\mathcal{PT}$-symmetry breaking.

The eigenfunctions of the above Hamiltonian also undergo characteristic changes as γ increases. A finite γ breaks the $\mathcal{P}$ symmetry of the Hamiltonian but, as long as $\gamma < \gamma_{\mathcal{PT}}$, the $\mathcal{PT}$ symmetry of the eigenfunction is preserved, so that $\psi_n^* = \pm\psi_{-n}$. This implies that the "dipole moment", $D = \sum_{n=-N}^{N} n|\psi_n|^2$, of an eigenstate is zero. For $\gamma > \gamma_{\mathcal{PT}}$ the eigenstates acquire a finite dipole moment. Below we shall use D as one of the signatures of the $\mathcal{PT}$-symmetry breaking.

The disordered $\mathcal{PT}$-symmetric chain.—We consider next a 1D disordered $\mathcal{PT}$-symmetric chain and demonstrate that under a disorder increase, the $\mathcal{PT}$-symmetric phase is gradually destroyed. For sufficiently strong disorder this phase shrinks to an exponentially narrow region, even for a comparatively small system.

The chain is described by the Hamiltonian of Eq. (1), where now all ε_n are random complex numbers, $\varepsilon_n = \beta_n + i\gamma_n$, with the constraint $\varepsilon_n = \varepsilon_{-n}^*$. One can envisage various possibilities for randomness, either in β_n or γ_n, or both. Below we present results for the case where β_n (for $n \geq 0$) are uniformly distributed on the interval $[-\beta/2; \beta/2]$ and $\gamma_n = \gamma = \mathrm{const}$ (for $n \geq 1$). It is crucial, for the $\mathcal{PT}$ symmetry, to implement the constraint $\beta_n = \beta_{-n}$ and $\gamma_n = -\gamma_{-n}$ (the latter implies $\gamma_0 = 0$). This constraint introduces a peculiar long-range correlation. To clarify the picture we start with the Hermitian limit $\gamma = 0$, and assume a long chain, such that even eigenstates in the band center are localized, i.e., their localization length is smaller than the system size $(2N + 1)$. Imagine for a moment that the chain is cut in half, at its center $n = 0$. Then a typical state in one half of the chain would be localized, with some localization length ξ, far away from $n = 0$, say, near site $l \gg 1$. This state has its counterpart in the other half of the chain, near site $-l$. In the full, connected chain this pair of states has an exponentially small overlap, at the center of the chain, yielding two real eigenstates of the entire chain. Each of these eigenstates (one symmetric, the other antisymmetric) has two peaks, near the sites $n = \pm l$. The energy splitting between the two eigenvalues is exponentially small, $\delta(l) \simeq e^{-2l/\xi}$, in complete analogy with the example of the two impurities in a perfect chain.

Thus, the eigenstates in a $\mathcal{P}$-symmetric disordered chain are organized into pairs (doublets) of energy splitting $\delta_1 < \delta_2 < \cdots$. The energy splitting between the two states of a doublet is exponentially small, while the energy separation between consecutive doublets is much larger, of the order of level spacing, Δ, in half of the chain. The pair associated with the minimum splitting, δ_1 is likely to originate from states localized far away from the origin of the chain ($n = 0$) and with energies at the band edges (small ξ). As γ is switched on, the eigenstates of each pair will initially preserve their $\mathcal{PT}$-symmetric structure [see Figs. 1(a) and 1(b)]. At $\gamma = \gamma_{\mathcal{PT}} \simeq \delta_1$, the two levels associated with δ_1 will cross, breaking the $\mathcal{PT}$ symmetry [see Fig. 2(a)]. As $\gamma > \gamma_{\mathcal{PT}}$ these modes cease to be eigenstates of the $\mathcal{PT}$ operator. Instead, the weight of each of them is gradually shifted towards one of the localization centers. An example of such pair associated with δ_1 is reported in Figs. 1(a) and 1(c). For larger γ the next doublet, with splitting δ_2 will come into play, creating a second pair of complex eigenvalues for $\gamma \simeq \delta_2$ [see Fig. 2(a)], etc.

The above qualitative considerations apply to a single, realization of the random potential. A full theory must be formulated in statistical terms and deal with probability distributions. For instance, the critical value $\gamma_{\mathcal{PT}}$ at which the $\mathcal{PT}$ symmetry is broken, fluctuates from one realization to another and the appropriate question pertains to the distribution $P(\gamma_{\mathcal{PT}})$. As has been argued above [see inset of Fig. 2(a)], in the strong disorder regime $\gamma_{\mathcal{PT}} \approx \delta_1$, and thus the problem reduces to the study of the distribution $\mathcal{P}(\delta_1)$. There are several sources of fluctuations in δ_1: fluctuations in the position and energy of the relevant localized states, as well as what can be termed "fluctuations in the wave functions". By this we mean that a localized wave function exhibits strong, log-normal fluc-

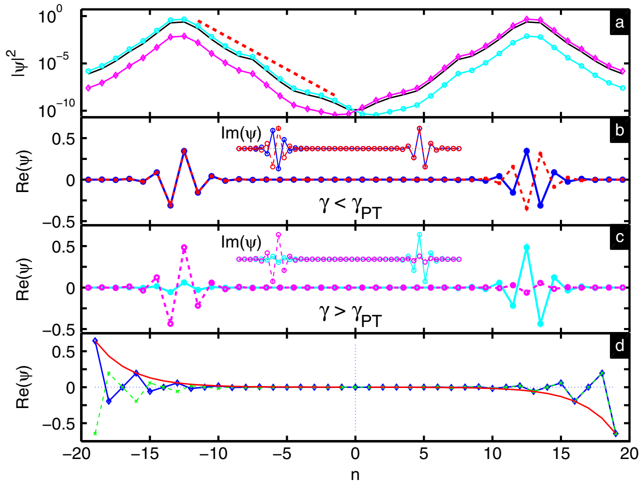

FIG. 1 (color online). Pairs of exponentially localized modes in a 1D chain with $\mathcal{PT}$-symmetric disorder [(a)–(c)] and surface states in a $\mathcal{PT}$-symmetric periodic chain (d) yielding the smallest energy splittings δ_1 for $\beta = 3$. For $\gamma < \gamma_{\mathcal{PT}}$ (b) the two eigenfunctions [blue (dark gray) and red (medium gray); the imaginary part is shown in the inset] are complex but $\mathcal{PT}$ symmetric and their absolute values remains equal and symmetric [coinciding on the black line in (a)]. For $\gamma > \gamma_{\mathcal{PT}}$ (c) the eigenfunctions (magenta and cyan) are no longer $\mathcal{PT}$ symmetric and shift their weight towards separate sides of the chain (a). (d) Surface states [blue (dark gray) and green (light gray)] showing exponential localization. The red (medium gray) dashed [solid] lines in (a) [(d)] are guides to the eye, pointing out the exponential localization.

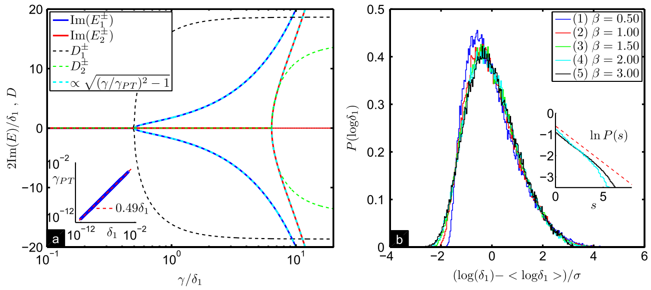

FIG. 2 (color online). (a) Bifurcations for the dipole moment D and for the imaginary part ImE of energy levels for a $\mathcal{PT}$-symmetric 1D chain with $\gamma = \mathrm{const}$ and β_n given by a box distribution $[-\frac{\beta}{2}; \frac{\beta}{2}]$ for $N/\xi \gg 1$. The first two bifurcations (corresponding to splittings δ_1 and δ_2) are shown. The square-root behavior at the branching point (see text) is indicated with dashed cyan lines (on top of the ImE lines). Inset: a linear relation between δ_1 and $\gamma_{\mathcal{PT}}$ is evident for almost 10 orders of magnitude. (b) Distribution $\mathcal{P}(x)$ of $x \equiv (\log\delta_1 - \langle\log\delta_1\rangle)/\sigma$ (σ is the standard deviation) for various disorder strengths β. In the limit of large β the distribution becomes log-normal. Inset: The distribution $\mathcal{P}(s)$ of $s \equiv \log(\delta_2) - \log(\delta_1)$ reported in a semilogarithmic plot. A Poisson distribution is approached as β increases.

tuations in its "tails", i.e., sufficiently far from its localization center [12]. This latter source of fluctuations appears to be the dominant one and it immediately yields a log-normal distribution for δ_1 [see Fig. 2(b)], since δ_1 is proportional to the overlap integral between a pair of widely separated and strongly localized states.

The aforementioned scenario of bifurcations, i.e., of the consecutive branching of pairs of eigenvalues into the complex plane, can also be made more quantitative. Consider, the separation (on the γ axis) between the first bifurcation ($\gamma = \gamma_{\mathcal{PT}}$) and the next one. This separation, $\delta\gamma$, is proportional to $(\delta_2 - \delta_1)$. Assuming that localized states, associated with the doublets, are randomly located, and ignoring for the moment fluctuations in the energy of these states, one immediately obtains a Poisson distribution for $s \equiv \log\delta_2 - \log\delta_1$, with the average spacing $2/\xi$ on the $\log\gamma$ scale between the bifurcation points [see inset of Fig. 2(b)]. This result, with ξ being replaced by an appropriate average, remains valid also when we account for the energy fluctuations.

Our considerations can be extended to the $N/\xi \ll 1$ limit, when the states responsible for δ_1 are extended over the entire chain. In this case the picture of doublets with exponentially small energy splittings is not valid and $\gamma_{\mathcal{PT}}$ becomes of the order of the minimal level spacing, $\Delta_{\min}$, in the corresponding Hermitian problem. This statement follows from perturbation theory, with respect to γ. The unperturbed (i.e., $\gamma = 0$) energy levels are real, and are separated by intervals of order $1/N^2$ at the band edges (at the center of the band the separation is of order $1/N$), so that $\Delta_{\min} \simeq 1/N^2$. Finite γ leads to level shifts proportional to γ^2 (the first order correction vanishes due to $\mathcal{PT}$ symmetry) and for $\gamma = \gamma_{\mathcal{PT}} \simeq \Delta_{\min}$ the perturbation theory breaks down, signaling level crossing and the appearance of the first pair of complex eigenvalues. Thus, the energy scale for the $\mathcal{PT}$ threshold in the $N/\xi \ll 1$ limit ($\gamma_{\mathcal{PT}} \simeq 1/N^2$) widely differs from that for $N/\xi \gg 1$ ($\gamma_{\mathcal{PT}} \simeq e^{-2N/\xi}$). However, the "bifurcation scenario," with characteristic square-root branches for the complex eigenvalues, holds in both cases (again, with the completely different energy scale for the intervals between the consecutive bifurcations). Our numerical results presented in Fig. 3(a) are in perfect agreement with these considerations.

We study now the distribution $P(\gamma_{\mathcal{PT}})$ in the limit $N/\xi \ll 1$. We invoke perturbation theory with respect to the perfect lattice. The perturbative scenario, indicates that weak disorder will cause a small shift of the energy levels. Thus the new level spacing becomes $\Delta_{\min} \pm \Delta\delta$, where the sign $+$ $(-)$ refers to the upper (lower) band edge. Regardless of the sign of $\Delta\delta$, the minimal level spacing (in first order perturbation theory) is

$$\delta = \Delta_{\min} - |\Delta\delta| \equiv \Delta_{\min} - \frac{4}{2N+1}\left|\sum_{n=1}^{N} A_n\beta_n\right|, \quad (4)$$

with the coefficients $A_n = \sin(\frac{\pi(n+N)}{2N+1})\sin(\frac{3\pi(n+N)}{2N+1})$. If β_n were Gaussian distributed, it would be immediately clear that the distribution of δ, $P(\delta)$, is a semi-Gaussian for

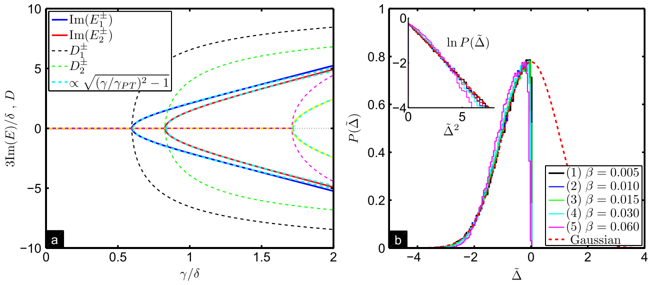

FIG. 3 (color online). (a) Same as in Fig. 2(a) but now for "weak" disorder $N/\xi \ll 1$. The same bifurcation scenario is observed. (b) The distribution $\mathcal{P}(\tilde\Delta)$ of the rescaled minimal energy split $\tilde\Delta = \Delta\delta/\sigma$ (σ is the standard deviation), for various disordered strengths, all of them being in the weak localized regime. The distribution has an upper cutoff at $\tilde\Delta = 0$. A Gaussian distribution is shown also by a red dashed line. Inset: The same data in a semilogarithmic manner vs $\tilde\Delta^2$.

$\delta < \Delta_{\min}$. This should remain approximately true also for the box distribution, employed in our numerics, if the number of terms in the sum is sufficiently large. Figure 3(b) confirms this expectation.

Periodic $\mathcal{PT}$-symmetric potentials.—Let us briefly address the problem of $\mathcal{PT}$-symmetry breaking for a periodic complex potential. This question has been raised in [6], for a potential $V(x) = 4(\cos^2 x + iV_0 \sin 2x)$, where it was argued that the critical value, of the $\mathcal{PT}$ threshold was $V_0^{\rm th} = 1/2$. The presence of the real part, $A\cos^2 x$ ($A \neq 0$), is crucial for this result. Indeed, it was pointed out in [6] that a purely imaginary periodic potential, $V(x) = iV_0 \sin^{2N+1}(x)$, treated in Ref. [2], has "zero $\mathcal{PT}$ threshold", i.e., it cannot have an entirely real spectrum. Another example of a periodic potential with zero $\mathcal{PT}$ threshold was provided in Ref. [3].

In a periodic system of finite extent one usually encounters localized states (surface states) at the boundaries of the sample [13]. We have found that these states, which were not addressed in Ref. [6], are crucial for the correct evaluation of the $\mathcal{PT}$ threshold in a finite system. We illustrate the importance of the surface states by taking the example of a tight binding periodic potential, with three sites per unit cell. The Hamiltonian is still that of Eq. (1), but with on-site energies: $\varepsilon_{n=3l} = 0$; $\varepsilon_{n=3l\pm1} = \beta \pm i\gamma$ where $l = 0, \pm1, \pm2, \ldots, \pm N/3$. For $\gamma = 0$ (and $N \to \infty$) the spectrum displays two energy gaps, whose width (for $\beta \ll 1$) is $2\beta/3$. The existence of gaps suggests, in analogy with [6], that the $\mathcal{PT}$-symmetric phase in this model will exhibit some robustness, as long as γ is small ($\gamma \ll \beta$). This expectation, however, is not born out due to the surface states. For $\gamma = 0$ there is a pair of surface states, exponentially decaying away from the sites $\pm N$. For large but finite N these surface states have a small overlap near the center of the chain $n = 0$, yielding a doublet with an exponentially small energy splitting. In complete analogy with the two-impurity problem, already an exponentially small $\gamma \simeq e^{-\beta N}$ will therefore break the $\mathcal{PT}$ symmetry. This example shows that the $N \to \infty$ limit can be quite subtle, as far as the $\mathcal{PT}$ threshold is concerned. If one starts directly with $N = \infty$, one obtains a finite $\mathcal{PT}$ threshold, $\gamma_{\mathcal{PT}} \simeq \beta$. If one starts, however, with a large but finite N and then takes the $N \to \infty$ limit (which is the correct physical limit), then one ends up with $\gamma_{\mathcal{PT}} = 0$, due to the existence of surface states.

Conclusion.—We have studied a 1D $\mathcal{PT}$-symmetric chain with disorder. The $\mathcal{PT}$-symmetric phase turns out to be very fragile. For a sufficiently long chain, this phase exists only for exponentially small values of the imaginary part of the potential $\gamma_{\mathcal{PT}} \simeq e^{-N/\xi}$ beyond which the $\mathcal{PT}$ symmetry is broken (here N and ξ are the system size and the localization length, respectively). Our model can be extended in various directions. For instance, we have checked that our main results do not change if randomness is introduced in both the real and the imaginary parts of the potential. We have also briefly discussed the periodic $\mathcal{PT}$ potential and pointed out the importance of surface states in breaking $\mathcal{PT}$ symmetry. Our main conclusion is within a generic 1D system which supports localized modes, the threshold for $\mathcal{PT}$-symmetry breaking is exponentially approaching zero with increasing size.

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