Coarsening of Faraday Heaps: Experiment, Simulation, and Theory

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When a layer of granular material is vertically shaken, the surface spontaneously breaks up in a landscape of small Faraday heaps that merge into larger ones on an ever increasing time scale. This coarsening process is studied in a linear setup, for which the average life span of the transient state with N Faraday heaps is shown to scale as N^{-3}. We describe this process by a set of differential equations for the peak positions; the calculated evolution of the landscape is in excellent agreement with both the experiments and simulations. The same model explains the observational fact that the number of heaps towards the end of the process decreases approximately as $N(t) \propto t^{-1/2}$.

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Introduction.—When a bed of fine dry sand is vertically vibrated or tapped, its initially flat surface turns into a landscape of small heaps, which in the course of time coarsen into larger ones. This phenomenon, known as Faraday heaping, is one of the most celebrated and beautiful examples of the effect of air on granular matter [1–3] and in a broader context, provides a prime example of spontaneous pattern formation in a dynamical system outside of equilibrium [4].

The dynamic stabilization of a single heap (last snapshot of Fig. 1) is well understood: The outward avalanches in the upper layers are balanced by the inward motion of the deeper layers (induced by the airflow through the vibrating bed [2,3]), together forming the convective flow of particles known as Faraday circulation. By contrast, the merging of small heaps into larger ones—the coarsening process (Fig. 1)—is much less understood, and quantitative experiments have been scarce [5,6].

In the present study we introduce a model for the coarsening behavior, validated by experiments and detailed numerical simulations. This threefold approach leads to the identification of the average life span τ_N of the N-heap state as the proper coarsening quantity. It is proven to scale, in our 1D setup, as $\tau_N \propto N^{-3}$.

Experiments.—A glass box of dimensions $L \times H \times D = 300 \times 100 \times 2.1$ mm^3 is vertically vibrated using a sinusoidal driving with frequency $f = 6.25$ Hz and amplitude $a = 10$ mm. The box contains 19.44 grams of spherical glass particles ($\rho = 2500$ kg/m^3) with an average diameter of $d = 0.5$ mm; i.e., the height of the granular bed in rest is about 31 particle diameters. The above choice of parameters means that we operate at a dimensionless acceleration $\Gamma = a(2\pi f)^2/g = 1.6$, so that the bed detaches from the vibrating bottom during part of the driving cycle [7]. This is necessary, since the heaping effect relies on the air flowing into (and out of) the void between bed and bottom, with the bed acting as a porous piston [3]. In addition, the dimensionless energy input $E = (a2\pi f)^2/(gd) = \Gamma a/d$ must be sufficiently high in order to sustain the convective circulation of grains within the heaps [3,5]. Our choice of $a/d = 20$ gives $E = 36$, which lies well above the required threshold value of $E \approx 2.0 + 1.26a/d = 27$ [5] and thus guarantees a smooth coarsening process.

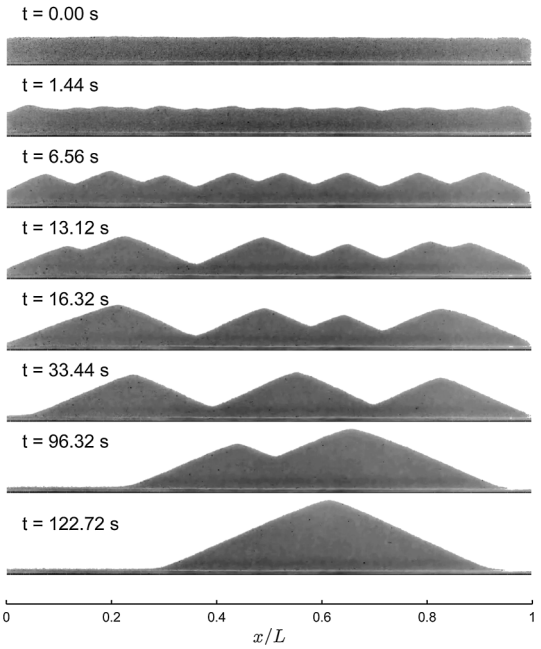

FIG. 1. Coarsening of a vertically vibrated 1D granular bed, as recorded in our experiments. It takes roughly two minutes to evolve from a flat landscape to a single Faraday heap. Every image is taken at the same point during the vibration cycle, when the container moves upward and the bed is pressed against the floor.

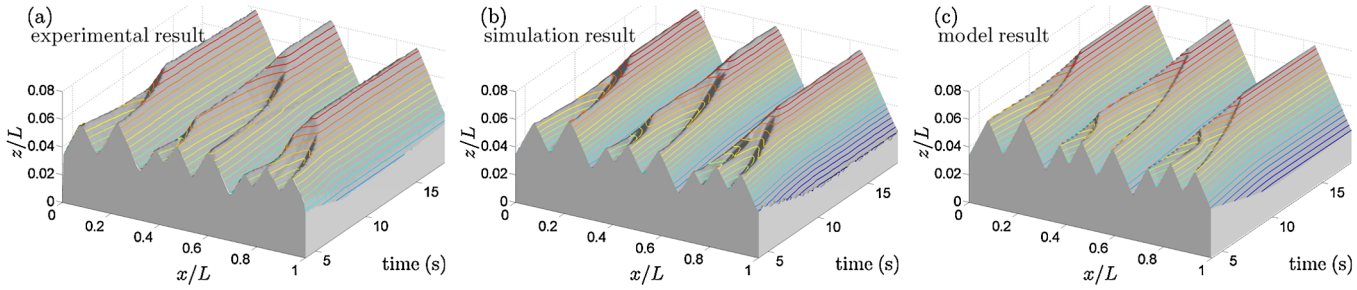

FIG. 2 (color online). Evolution of the heap pattern from $t = 4$ s to 18 s obtained by (a) experiment, (b) simulation, and (c) the analytical model. A stroboscopic video of the evolving heap pattern as obtained by experiment, simulation, and model is supplied in the supplementary material in [11].

Where Fig. 1 shows snapshots from a typical experimental run, Fig. 2(a) contains the time evolution for a second run. In order to get sufficient statistics to determine the mean life span of the N-heap state, 19 runs were performed. Figure 3 shows the number of heaps $N(t)$ (a decreasing step function) for all 19 experiments [8]. The inset shows the averaged data on a log-log scale, suggesting that $N(t) \propto t^{-\beta}$ with β close to 0.5. We do not find the exponential decay reported by van Doorn and Behringer [5]. Presumably the exponential behavior is a critical case, since it was only observed in experiments for which the energy input E was around the value that is minimally required for heaping. When E exceeded this threshold, as in our case, also van Doorn and Behringer found a clear deviation from exponential decay.

Numerical simulations.—We performed numerical simulations on the same system. Our code combines granular dynamics (GD) and computational fluid dynamics (CFD) [9]: The GD part calculates the particle trajectories from Newton's law, with the particle-particle interactions being given by a 3D soft sphere collision model including tangential friction, while the CFD part evaluates the full Navier-Stokes equations for the interstitial air by a finite difference method. The two parts of the code are coupled to account for the effect of the air on the particles, and vice versa. The position and height of the peaks at $t = 4$ s in the experiment of Fig. 2(a) are used to create the starting condition for the numerical simulation [Fig. 2(b)]. The excellent correspondence confirms that any unwanted side-effects in the experimental setup (due to, e.g., misalignment, humidity, or static electricity) have been successfully kept to a minimum. Apart from the heap patterns, the simulations can provide detailed information that cannot be readily obtained from the experiments, and which will presently be used in setting up the analytical model.

Analytical model.—We will derive equations for the time evolution of the position $x_i(t)$ and height $z_i(t)$ of a typical heap (see Fig. 4), which are then combined to determine the evolution of the whole system. This model is based on observations that have been discussed in detail in [3], namely, the constancy of the slope angle, the fact that the horizontal motion of grains within a heap results from the pressure gradient just below its slopes, and the observation that a heap as a whole moves due to an asymmetry between its left and right slope lengths.

A key ingredient of our model is that the slope angle is the same for all heaps ($\alpha = 18.5°$ in our experiments) and remains constant during the entire coarsening process [3]; see Figs. 1, 2(a), and 2(b).

A second ingredient (from the simulations) is that when the bed detaches from the vibrating plate, the lines of constant air pressure run parallel to the slopes just below the surface, whereas deeper inside the heap the equal pressure lines flatten out [3]. So below a certain depth h the horizontal component of the air drag becomes negligible, which means that the total horizontal drag force F_x on the left part of the heap scales with the slope length l. This force acts during a small fraction of the driving cycle δt_1 (see [3]) and as a result, the particles in the left dark grey triangle in Fig. 4 (representing a mass $m_l \propto l^2$) are set in motion. Analogously, for the right side of the heap the force is proportional to r and will be in the negative x direction. Using conservation of momentum, the total ef-

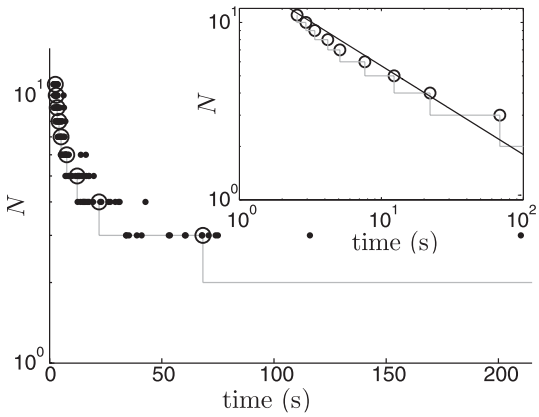

FIG. 3. Experiment: Number of heaps N as a function of time. The black dots are the measured times when the N-heap state gives way to the $(N-1)$-state; the open circles represent the average over all 19 experimental runs. No data for $N = 2$ are shown [8]. Inset: The averaged data on a doubly logarithmic scale reveals no clear scaling behavior for $N(t)$, which is explained in the text. The solid line corresponds to the approximation Eq. (7).

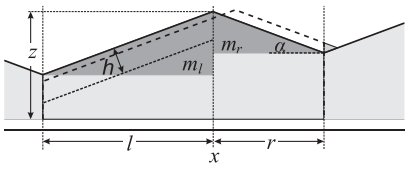

FIG. 4. Part of a typical heap pattern, indicating the key parameters used in the coarsening model. The dashed profile indicates the position of the heap after one time step dt.

fective mass ($\propto l^2 + r^2$) acquires a horizontal velocity $u_x \propto \delta t_1 (l - r)/(l^2 + r^2)$. This velocity is maintained during a considerable fraction of the vibration cycle δt_2, until the heap collides again with the vibrating plate [3]. So during the period of each vibration cycle, $\Delta t = 1/f$, the top of the heap will be displaced over a distance $\Delta x = u_x \delta t_2$. Since δt_1 and δt_2 do not change during a single realization of the experiment, the time rate of change of the horizontal position x of the top of the heap (on a time scale much larger than the duration of a cycle) is thus given by

$$\frac{dx}{dt} = C \frac{l - r}{l^2 + r^2}, \tag{1}$$

with C a constant that can be determined from experimental or simulation data (cf. Fig. 5).

To derive an analogous equation for the change of the heap height z, we note that the mass in a heap remains the same—in good approximation—from one cycle to the next. This is because the main flow of material is directed away from the valleys, towards the center of each heap, so there is hardly any mass being exchanged through the vertical lines at the valley positions. Thus the total area contained in a heap at time t, $A(t) = z(l + r) - \frac{1}{2} \times \tan\alpha (l^2 + r^2)$, will still be the same at time $t + \Delta t$, indicated by the dashed profile in Fig. 4. Setting $A(t) = A(t + \Delta t)$ [the latter with $z \to z + \Delta z$, $l \to l + \Delta x$, and $r \to r - \Delta x$], we obtain in the limit of Δx, $\Delta z \to 0$ the desired equation for the height z:

$$\frac{dz}{dt} = \tan\alpha \frac{l - r}{l + r} \frac{dx}{dt} = C \frac{\tan\alpha (l - r)^2}{(l + r)(l^2 + r^2)}. \tag{2}$$

To complete the model, we use the fact that in each cycle, after the heap has shifted, avalanches relax the slope angles to the value α again. This leads to a relocation of the ith valley (between peak $\{x_i, z_i\}$ and $\{x_{i+1}, z_{i+1}\}$) such that its horizontal position $\check{x}_i$ is given by

$$\check{x}_i = \frac{1}{2}(x_{i+1} - x_i) - \frac{1}{2 \tan\alpha}(z_{i+1} - z_i). \tag{3}$$

The change of area due to this relocation of the valleys (the small grey triangle in Fig. 4) is of second order in dx and therefore in dt. It thus vanishes in the limit $dt \to 0$ and the total mass in the system is conserved.

The model contains one parameter C, which sets the absolute time scale of the coarsening process. It depends on particle size and density, gravity, and viscosity, as well as on the vibration parameters Γ and a/d, keeping pace with the Faraday circulation velocity (see [3]). However, keeping the above parameters fixed, C is simply a constant and its value can be determined as in Fig. 5. With C given, one can numerically solve the model equations; we use the experimental peak positions $x_i(t)$, $z_i(t)$ at $t = 4$ s to define our initial condition. The solution [Fig. 2(c)] is seen to match the experimental and simulated patterns very well. We conclude that our model accurately captures the essential features of the process.

Mean life span of the N-heap state.—The model is ideally suited to study the scaling behavior of the coarsening process. It allows us to start with an arbitrarily large number of heaps (which in experiment would require a forbiddingly long box) and moreover, to perform thousands of different realizations to improve the statistics. In Fig. 6 we show both the average number of heaps $N(t)$ as a function of time (inset) and the mean life span τ_N of the N-heap state as a function of N. The latter is defined as the average time that elapses from the moment the system switches from $N + 1$ to N heaps until the switch from N to $N - 1$ heaps. The black circles represent the average

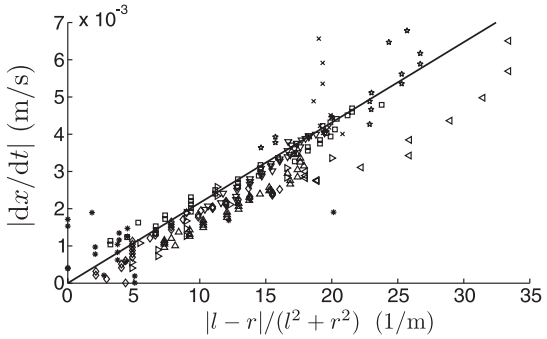

FIG. 5. Validation of Eq. (1) from the simulation results in Fig. 2(b). The slope of the fitted line gives the factor $C = 0.2 \times 10^{-3}$ m^2/s. The data in this figure are taken from the eight heaps in Fig. 2(b), with each heap being indicated by a different marker.

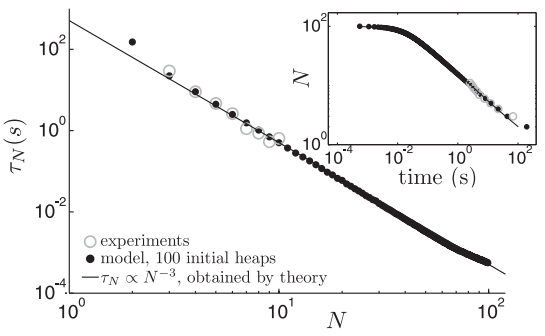

FIG. 6. Mean life span of the N heap state τ_N as a function of the number of heaps N. The grey circles indicate the experimental data. The black circles indicate the data averaged over 10 000 runs of the model starting with 100 initial heaps. Inset: Number of heaps as a function of time (cf. Fig. 3).

over 10 000 runs of the model, each run starting out from $N = 100$ heaps with slope lengths that are uniformly distributed between 0 and $L/100$ [10]. For τ_N we see a clear power-law scaling $\tau_N \propto N^{-3}$ over the full two decades of N, and once more an excellent agreement between model and experiment in the final decade $N = 10, \ldots, 3$. The average number of heaps however (inset) does *not* exhibit global power-law scaling. So not $N(t)$, but τ_N is the natural quantity to analyze from a theoretical point of view.

Explanation of the scaling law.—How does $\tau_N \propto N^{-3}$ follow from the model? To answer this, we rewrite Eqs. (1)–(3) in terms of the previously introduced left and right slope lengths $l_i = x_i - \check{x}_{i-1}$ and $r_i = \check{x}_i - x_i$. After some algebra this leads to

$$\begin{aligned} dl_i/dt &= f(l_i, r_i) - f(l_{i-1}, r_{-1}), \\ dr_i/dt &= f(r_i, l_i) - f(r_{i+1}, l_{i+1}), \end{aligned} \tag{4}$$

where the function f is given by

$$f(u, v) = C\frac{u(u - v)}{(u^2 + v^2)(u + v)}. \tag{5}$$

These equations can be nondimensionalized as follows: We divide all lengths l_i and r_i by the average slope length in the N-heap state ($= L/2N$, in the absence of depletion effects [8]): $\tilde{l}_i = 2l_iN/L$ and $\tilde{r}_i = 2r_iN/L$. Defining $\tilde{f} = f/C$ we then arrive at

$$\frac{d\tilde{l}_i}{d\tilde{t}} = \tilde{f}(\tilde{l}_i, \tilde{r}_i) - \tilde{f}(\tilde{l}_{i-1}, \tilde{r}_{i-1}) \tag{6}$$

(and similarly for $d\tilde{r}_i/d\tilde{t}$), in which the dimensionless time coordinate $\tilde{t}$ must be defined as $\tilde{t} = 4N^2Ct/L^2$.

Now we focus on a single heap i in a N-heap state ($i = 1, \ldots, N$) and compute its time-evolution with Eq. (6) until it merges with one of its two neighbors. This yields a dimensionless lifetime $\tilde{T}_i$ of the heap, which can subsequently be translated to its dimensional value: $T_i = \tilde{T}_iL^2/(4CN^2)$. So the lifetime T_i of an arbitrary heap i in the N-heap state scales as $1/N^2$. The life span of this N-heap state is equal to the *shortest* T_i (since the first heap that merges terminates the N-heap state) and therefore decays faster than $1/N^2$, because the statistical minimum of a sample of N values naturally decreases with N. More specifically, when N is not too small, the merging events can be considered as independent, so that the set of T_i's for the N-heap state obeys an exponential distribution. One of the properties of such a distribution (see the supplementary material [11]) is that the average minimum value, i.e., the minimum value in a set of N numbers taken randomly from an exponential distribution, decreases with the sample length as $1/N$. Hence the mean life span τ_N scales as $1/N^3$, which explains the observed scaling behavior.

The above analysis also shows that, for the current problem, $N(t)$ is a more intricate quantity than τ_N: The total elapsed time $t(N)$ at the end of the N-heap state equals the sum of the τ_N of all states that lie between the initial number of heaps N_{init} and N. For small N we approximate

$$t(N) = \sum_{N'=N}^{N_{\text{init}}} \tau_{N'} \approx \int_N^{N_{\text{init}}} \tau_{N'} dN' \propto \int_N^{N_{\text{init}}} \frac{dN'}{N'^3}, \tag{7}$$

so that $t(N)$ scales as N^{-2} for $N \ll N_{\text{init}}$ or vice versa, $N(t) \propto t^{-1/2}$. This explains the behavior found in Fig. 3, as well as in the data for small N in the inset of Fig. 6.

Conclusion.—The essence of the coarsening of Faraday heaps has been captured in a simple system of differential equations. The model gives results that are in excellent quantitative agreement with both the experiment and simulations. Combining the three approaches, we have shown that τ_N, the mean life span of the N-heap state, scales as N^{-3} during the entire coarsening process. Furthermore, towards the end of the process the number of heaps decreases approximately as $N(t) \propto t^{-1/2}$.

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[1] M. Faraday, Phil. Trans. R. Soc. London **121**, 299 (1831); C. Laroche, S. Douady, and S. Fauve, J. Phys. (Paris) **50**, 699 (1989); H. K. Pak, E. van Doorn, and R. P. Behringer, Phys. Rev. Lett. **74**, 4643 (1995); H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Rev. Mod. Phys. **68**, 1259 (1996); R. P. Behringer, E. van Doorn, R. R. Hartley, and H. K. Pak, Granular Matter **4**, 9 (2002).
[2] B. Thomas and A. M. Squires, Phys. Rev. Lett. **81**, 574 (1998).
[3] H. J. van Gerner *et al.*, Phys. Rev. E **76**, 051305 (2007).
[4] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993); I. S. Aranson and L. S. Tsimring, Rev. Mod. Phys. **78**, 641 (2006).
[5] E. van Doorn and R. P. Behringer, Phys. Lett. A **235**, 469 (1997).
[6] T. Shinbrot, Granular Matter **1**, 145 (1998).
[7] P. Eshuis *et al.*, Phys. Fluids **19**, 123301 (2007).
[8] Note that in our experiments the box becomes depleted towards the end of the process, so there are areas where hardly any material is present anymore (see Fig. 1). We only considered data free of depletion effects, meaning that for $N \leq 2$ no experimental data were used at all.
[9] M. A. van der Hoef *et al.*, Adv. Chem. Eng. **31**, 65 (2006).
[10] If we start from a narrow distribution of heaps, the initial mean life spans are larger than predicted by the power law, but soon the distribution broadens and the life spans τ_N converge to the straight line of Fig. 6.
[11] See EPAPS Document No. E-PRLTAO-103-053929 for supplementary material. For more information on EPAPS, see http://www.aip.org/pubservs/epaps.html.