

Renormalized Resonance Quartets in Dispersive Wave Turbulence

Wonjung Lee,¹ Gregor Kovačič,² and David Cai^{1,3,*}

¹*Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012, USA*

²*Mathematical Sciences Department, Rensselaer Polytechnic Institute, 110 8th Street, Troy, New York 12180, USA*

³*Mathematics Department, Shanghai Jiao Tong University, Shanghai 200240, China*

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Using the (1 + 1)D Majda-McLaughlin-Tabak model as an example, we present an extension of the wave turbulence (WT) theory to systems with strong nonlinearities. We demonstrate that nonlinear wave interactions renormalize the dynamics, leading to (i) a possible destruction of scaling structures in the bare wave systems and a drastic deformation of the resonant manifold even at weak nonlinearities, and (ii) creation of nonlinear resonance quartets in wave systems for which there would be no resonances as predicted by the linear dispersion relation. Finally, we derive an effective WT kinetic equation and show that our prediction of the renormalized Rayleigh-Jeans distribution is in excellent agreement with the simulation of the full wave system in equilibrium.

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For many wave phenomena, due to their inherent complexity and turbulent nature, statistical ensembles rather than individual wave trajectories render the natural observables. In numerous branches of physics, including surface waves, capillary waves, internal waves, waves on liquid hydrogen, Alfvén and Langmuir waves in plasmas, and turbulence in nonlinear optics, wave turbulence (WT) [1–3] arises through interactions of weakly nonlinear resonant waves in a dispersive medium. In contrast to strong turbulence in incompressible fluids, the weak nonlinearity in wave interactions potentially allows for a systematic treatment of WT [3]. The resulting kinetic equation in WT theory captures the time evolution of wave action [3]. In addition to the equilibrium Rayleigh-Jeans (RJ) distribution, there are Zakharov-Kolmogorov stationary solutions [3] to the kinetic equation for homogeneous, scale-invariant wave systems, which capture the direct and inverse cascades of wave excitations. These were believed to be universal (i.e., independent of the details of driving and damping) nonequilibrium spectra in an inertial range where neither driving nor damping exists.

Invoking random phase approximation (RPA), near Gaussianity in wave statistics (so no coherent structures) and resonant wave-wave interactions, WT theory was formally developed for describing the long-time statistical behavior of waves. Yet a major question remains, namely, how well it can describe real wave systems. Many studies attempted to verify the results of WT theory using direct numerical simulations of the underlying wave equations, but careful examination of the validity conditions of WT theory is further needed, in particular, on questions of what happens if any of the assumptions leading to WT theory are violated. This requires careful analysis of the related wave and (integro-differential) kinetic equations, with accurate simulations for precise statistical convergence. For (1 + 1)D dispersive waves, this was carried out by intro-

ducing the MMT model [4], whose Hamiltonian in the Fourier space is given by $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4$,

$$\mathcal{H}_2 = \int \omega_k |a_k|^2 dk, \quad (1)$$

$$\mathcal{H}_4 = \frac{1}{2} \int T_{1234} a_{k_1} a_{k_2} a_{k_3}^* a_{k_4}^* \delta(k_1 + k_2 - k_3 - k_4) dk_{1234},$$

with $\omega_k = |k|^\alpha$, $T_{1234} = |k_1 k_2 k_3 k_4|^{\beta/4}$, parameters $\alpha > 0$ and β , $dk_{1234} = dk_1 dk_2 dk_3 dk_4$ and $\delta(\cdot)$ denoting the Dirac delta function.

In this Letter, using the MMT model as a prototypical example, we show how nonlinearity strongly modifies the dynamics of resonance structure and discuss its consequences for the long-time dynamics of WT. Our study reveals (i) the scaling-structures in the bare Hamiltonian (1) can be destroyed and resonant manifolds are qualitatively modified by the renormalized dispersion relation, which is a generalization of weak turbulence results [5] to strong nonlinearities and (ii) nonlinear interactions can create resonances in wave systems whose bare dynamics has no resonance according to the linear dispersion relation. Finally, we extend the WT theory to include renormalized resonance dynamics.

The MMT model (1) is a prototypical example of a homogeneous, scale-invariant system that allows for four-wave resonances in one dimension in case of a concave dispersion law, i.e., $\alpha < 1$ (for which there are no three-wave resonances). Its canonical equation of motion is

$$i \frac{\partial a_k}{\partial t} = \frac{\delta \mathcal{H}}{\delta a_k^*}. \quad (2)$$

If $\alpha = 2$ and $\beta = 0$, it corresponds to the nonlinear Schrödinger equation (NLS) while the case of $\alpha = 1/2$, $\beta = 3$ mimics the scalings present in water waves. Numerical studies [4,6] of system (1) reveal self-similar,

often coexistent, spectra, including one apparently inconsistent with WT theory, as well as coherent structures, such as solitons, quasisolitons, and collapses, which greatly complicate the WT picture of the MMT system. Studies of this model thus show that, even in the weakly nonlinear limit, WT theory may not be able to capture fully the rich behavior of the nonlinear wave system [6,7].

Since the longtime statistical behavior of the nonlinear system is often controlled by resonances, WT theory focuses on the resonant wave interactions [3] determined by the linear dispersion relation $\omega_k = \omega(k)$:

$$\Delta_{k_3 k_4}^{k_1 k_2} \equiv k_1 + k_2 - k_3 - k_4 = 0, \quad (3a)$$

$$\Delta_{\omega_3 \omega_4}^{\omega_1 \omega_2} \equiv \omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} = 0. \quad (3b)$$

WT theory assumes that waves interact weakly, and thus, in equilibrium, give rise to the RJ distribution—a stationary solution of the kinetic equation, independent of the details of the nonlinearity [3]. However, nonlinear wave interactions tend to renormalize dispersion relations [8], which may have a strong impact on wave-wave interactions and resonant structures. Using the MMT system as a WT model, we investigate the consequences of dispersion renormalization for resonant wave interactions in both weakly and strongly nonlinear limits.

We first show that, in *equilibrium*, the Zwanzig-Mori (ZM) theory [9] can successfully describe how the dispersion relation is renormalized for long waves. This theory yields a generalized Langevin equation governing effective dynamics of slow observables. For a single dynamical variable $a_k(t)$, this exact Langevin equation is given by $\partial a_k(t)/\partial t = -i\Omega_k a_k(t) - \int_0^t \mathbf{K}(t-s)a_k(s)ds + F_k(t)$, where F_k is the random force related to the memory kernel \mathbf{K} by the fluctuation-dissipation theorem [9]. Using the equipartition theorem $\theta = \langle a_k^* \delta \mathcal{H} / \delta a_k^* \rangle$, where θ is the temperature of the MMT system, and $\langle \cdot \rangle$ denotes the average over the Gibbs measure $e^{-\mathcal{H}/\theta}$, we can show that the effective dispersion relation is

$$\begin{aligned} \Omega_k &= \frac{\theta}{\langle |a_k|^2 \rangle} \\ &= |k|^\alpha + |k|^{\beta/4} \\ &\quad \times \int |k_1 k_2 k_3|^{\beta/4} \frac{\langle a_{k_1} a_{k_2} a_{k_3}^* a_k^* \rangle}{\langle |a_k|^2 \rangle} \delta(\Delta_{k_3 k}^{k_1 k_2}) dk_{123}. \end{aligned} \quad (4)$$

The ZM projection formalism usually results in a linear, non-Markovian process. However, for a slow dynamical variable such as a long-wave mode, Markovian behavior results, and the renormalized dispersion relation Ω_k characterizes the temporal frequency of $a_k(t)$ [10]. For short waves, it is not clear that there is a time-scale separation among the linear dispersion, memory kernel, and random forcing; therefore, it would be difficult to interpret Ω_k as the oscillation frequency of a_k for high k 's. However, it will be seen below that, surprisingly, Ω_k accurately describes the oscillations of a_k for all k 's.

The renormalized ZM dispersion (4) further reduces to

$$\tilde{\Omega}_k \equiv |k|^\alpha + \left(2 \int |k'|^{\beta/2} \langle |a_{k'}|^2 \rangle dk' \right) |k|^{\beta/2} \quad (5)$$

by RPA. Via $\tilde{\Omega}_{k'} \doteq \theta / \langle |a_{k'}|^2 \rangle$, Eq. (5) becomes $\tilde{\Omega}_k = |k|^\alpha + \theta (2 \int |k'|^{\beta/2} \tilde{\Omega}_{k'}^{-1} dk') |k|^{\beta/2}$, from which $\tilde{\Omega}_k$ and θ can be determined after invoking the conservation of wave action, $\int |a_{k'}|^2 dk' \equiv \mathcal{N}$, i.e., $\int \theta / \tilde{\Omega}_{k'} dk' = \langle \mathcal{N} \rangle$, where \mathcal{N} is set by the initial condition. The connection between this renormalized dispersion and wave interactions can be seen by considering the collective effect of the trivial resonances, i.e., $k_1 = k_3$ or $k_1 = k_4$ in conditions (3). More precisely, the trivial resonant terms in \mathcal{H}_4 , $\mathcal{H}_4^{\text{tr}} \equiv 2 \int |k'|^{\beta/2} |k|^{\beta/2} |a_{k'}|^2 |a_k|^2 dk' dk$, can be approximated by $\mathcal{H}_q^{\text{eff}} = \int [(2 \int |k'|^{\beta/2} \langle |a_{k'}|^2 \rangle dk') |k|^{\beta/2}] |a_k|^2 dk$ via a mean-field argument that each a_k interacts effectively with the thermal background waves $\langle |a_{k'}|^2 \rangle$. The combination of $\mathcal{H}_q^{\text{eff}}$ and \mathcal{H}_2 yields an effective quadratic interaction $\mathcal{H}_2^{\text{eff}} \equiv \int [|k|^\alpha + (2 \int |k'|^{\beta/2} \langle |a_{k'}|^2 \rangle dk') |k|^{\beta/2}] |a_k|^2 dk$. Hence, the dispersion relation (5). Therefore, the longtime dynamics can be described by an *effective* Hamiltonian $\mathcal{H}^{\text{eff}} = \mathcal{H}_2^{\text{eff}} + \mathcal{H}_4 - \mathcal{H}_4^{\text{tr}}$, with $\mathcal{H}_4 - \mathcal{H}_4^{\text{tr}}$ representing the nonlinear interactions. This dispersion renormalization, arising from trivial resonant interactions, effectively weakens the averaged nonlinear interactions. Note that Eq. (5), which is not limited to weak nonlinearities, is a nonperturbative generalization of the perturbatively corrected dispersion relation for weak nonlinearities [5,11].

We now turn to the examination of our predictions (4) and (5). We numerically solve Eq. (2) [12,14] to obtain the spatiotemporal spectrum $|\hat{a}_k(\omega)|^2$ in equilibrium, where $\hat{a}_k(\omega)$ is the Fourier transform of $a_k(t)$. The peak locations, ω_k^{meas} , of $|\hat{a}_k(\omega)|^2$ can be viewed as the effective oscillation frequency of $a_k(t)$. Figure 1 displays the result for $\alpha = 1/2$ and $\beta = 6$, the inset for $\alpha = 2$ and $\beta = 0$ (NLS) where the correction to ω_k is an additive constant. In general, we find that, among the three dispersion relations, the ZM dispersion relation Ω_k agrees best with the measured ω_k^{meas} . We

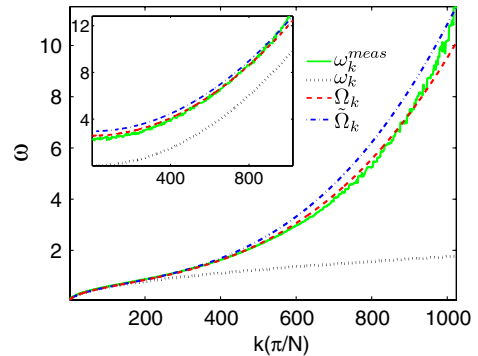


FIG. 1 (color online). Measured ω_k^{meas} , the bare $\omega_k = |k|^\alpha$, Eq. (4) and (5) with $\alpha = 1/2$, $\beta = 6$, are depicted as solid, dotted, dashed, and dashed-dotted lines, respectively. $N = 1024$. Inset: The same for NLS ($\alpha = 2$, $\beta = 0$).

stress that the RPA in $\tilde{\Omega}_k$ (5) is appealing in that it not only agrees well with numerical results, but also gives a clear, intuitive physical mechanism for the renormalization. It is important to point out that, unlike the Fermi-Pasta-Ulam chains [8], the renormalized wave frequency of the MMT system, in general, is not a simple rescaling of the bare $\omega_k = |k|^\alpha$. The case in Fig. 1 is characterized by the geometric shift from the overall concavity of the bare $\omega_k = |k|^{1/2}$ to the convexity of the renormalized curves at high k 's. This qualitative change of the dispersion relation takes place whenever $\alpha < 1$ and $\beta/2 > 1$, as seen in Eqs. (4) and (5), which generalizes, to strong nonlinearity, the corresponding results at weak nonlinearity [5]. It is important to note that the renormalization correction is $O(|k|^{\beta/2})$, which can always dominate over the bare dispersion relation $\omega_k = |k|^\alpha$ for large k 's if $\beta/2 > \alpha$, no matter how small the nonlinearity. Furthermore, except for $\beta = 2\alpha$, the scaling structures in the bare dynamics (1) are destroyed by renormalization even at weak nonlinearities, thus, giving rise to a new resonance manifold not determined by the original scaling symmetry, as discussed below.

The theoretic resonance structure (3) can be visualized by projecting

$$|\Delta_{\omega_3\omega_4}^{\omega_1\omega_2}| = |\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}| \quad (6)$$

on (k_1, k_2) with k_3 being fixed and k_4 from (3a). Figures 2(a)–2(c) display surface plots of (6) for the bare $\omega_k = |k|^\alpha$, and the renormalized $\omega_k = \Omega_k$ (note that using $\tilde{\Omega}_k$ gives similar results) for $\beta = 4$ [Fig. 2(b)] and $\beta = 8$ [Fig. 2(c)], respectively. In these figures, the resonance manifold determined by $|\Delta_{\omega_3\omega_4}^{\omega_1\omega_2}| \cong 0$, as signified by the dark strips, undergoes a deformation as β increases and the resonance structures determined by the renormalized Ω_k are clearly different from those by the bare ω_k . To approximate the MMT model, we use N Fourier modes, and move all to the first Brillouin zone. The resonances within the area in Fig. 2(a) bounded by the two dashed lines are system intrinsic, i.e., not caused by the periodicity of the finite system.

In the traditional WT theory, waves interact through resonances controlled by the bare ω_k . Here, we demonstrate a different picture. Since the resonances control the contribution of terms such as $a_{k_1} a_{k_2} a_{k_3}^* a_{k_4}^* \delta(\Delta_{k_3 k_4}^{k_1 k_2})$ in the longtime limit, we use the longtime average

$$\mathcal{A}_{k_3 k_4}^{k_1 k_2} \equiv \langle a_{k_1} a_{k_2} a_{k_3}^* a_{k_4}^* \tilde{\delta}(\Delta_{k_3 k_4}^{k_1 k_2}) \rangle$$

to reveal the resonance structures manifested in the dynamics (1). Here $\tilde{\delta}$ equals 1 if $\Delta_{k_3 k_4}^{k_1 k_2}$ is a multiple of N and 0 otherwise. This $\tilde{\delta}$, instead of the Dirac δ , is used to account for the discrete approximation of Eq. (2). For $\beta = 8$, $|\mathcal{A}_{k_3 k_4}^{k_1 k_2}|$ is displayed in Fig. 2(f), whose comparison with Fig. 2(c) reveals an excellent agreement between the locations of the peaks (dark strips) of the longtime average and the loci of the resonances (6) determined by the renormalized $\Omega_k^R \equiv \Omega_k$ or $\tilde{\Omega}_k$ (The WT theory would predict the

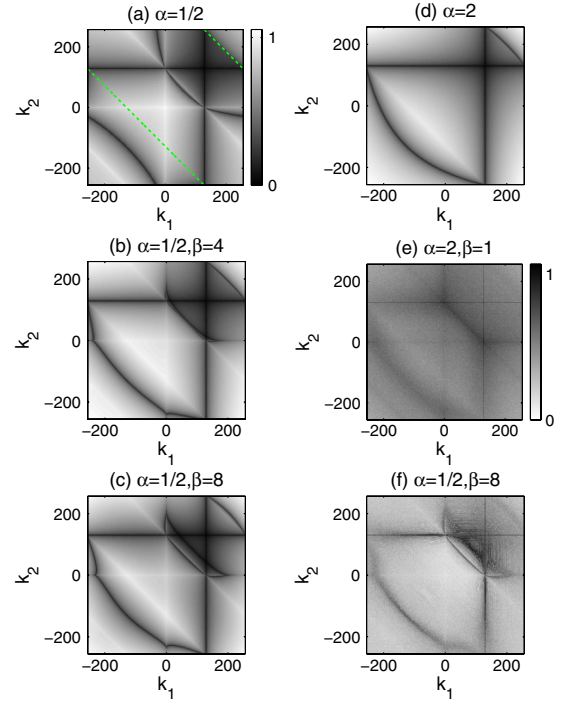


FIG. 2 (color online). The plot of the value of Eq. (6) using bare $\omega_k = |k|^\alpha$ (a),(d) or renormalized $\omega_k = \Omega_k^R$ (b),(c) vs $|\mathcal{A}_{k_3 k_4}^{k_1 k_2}|$ (e),(f). In all cases $k_3 = 128$ is used. k is in the unit of π/N , $N = 512$.

resonance structures as in Fig. 2(a) for these cases). The physical picture derived from these results is that wave resonances are renormalized and they are governed by the renormalized Ω_k^R . We note in passing that $\Omega_k^R = |k|^\alpha + \text{const}$ for NLS; therefore, its renormalized resonance structures should be the same as those predicted by WT theory, as is confirmed in our study.

We stress that both the nonlinearity parameter β and the linear frequency exponent α play important roles in Ω_k^R . In particular, if $\alpha > 1$ (for which there is no nontrivial four-wave resonances by $\omega_k = |k|^\alpha$), resonances controlled by Ω_k^R may arise if $0 < \beta/2 < 1$. Shown in Fig. 2(e) is such a result where new resonance structures for $\alpha = 2$, $\beta = 1$, are created. For comparison, the resonance structure (6) for $\omega_k = |k|^2$ is displayed in Fig. 2(d) which does not possess new resonant strips appearing in Fig. 2(e). This result shows that the nonlinearity renormalizes the linear dispersion relation to modify the resonance manifold, thereby creating new resonant interactions even when there would be no bare resonance as dictated by the linear dispersion relation. We note that there is a surprising similarity in the resonance structure between Figs. 2(b) and 2(e). This similarity arises because both Ω_k^R have the asymptotic form of $c_1 |k|^{1/2} + c_2 |k|^2$. The resonance structure of Fig. 2(b) is in a weak turbulence regime while Fig. 2(d) is in a strong nonlinear regime with $\mathcal{H}_4/\mathcal{H}_2 \sim 1$.

The classical kinetic equation of WT theory [3] cannot be used to find the spectra with strong nonlinearities. Here,

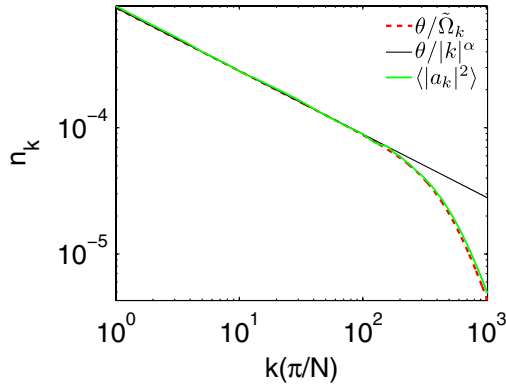


FIG. 3 (color online). Wave spectrum $\theta/\tilde{\Omega}_k$ (dashed) vs the WT prediction $\theta/|k|^\alpha$ (solid straight) with $\alpha = 1/2$, $\beta = 6$. Solid curve is measured $n_k = \langle |a_k|^2 \rangle$. $N = 1024$.

we develop a WT-like theory for this regime based on the frequency renormalization derived above. Approximating $\langle |a_k(t)|^2 \rangle$ by $n_k(t) \equiv \langle |a_k(t)|^2 \rangle_{\text{eff}}$, averaged over the Gibbs measure with the effective Hamiltonian $\mathcal{H}^{\text{eff}} = \mathcal{H}_2^{\text{eff}} + \mathcal{H}_4 - \mathcal{H}_4^{\text{tr}}$, we can derive the following effective kinetic equation for $n_k(t)$:

$$\frac{\partial n_k(t)}{\partial t} = 4\pi \int \tilde{T}_{123k} U_{123k} \delta(\Delta_{\tilde{\Omega}_3 \tilde{\Omega}_k}^{\tilde{\Omega}_1 \tilde{\Omega}_2}) \delta(\Delta_{k_3 k}^{k_1 k_2}) dk_{123}, \quad (7)$$

where $U_{123k} = n_1 n_2 n_3 n_k (n_k^{-1} + n_3^{-1} - n_2^{-1} - n_1^{-1})$ and the interaction tensor $\tilde{T}_{123k} = T_{123k}$ if $k_1 \neq k_2$ and $k_3 \neq k$, and 0 otherwise. We immediately find $n_k = \theta/\tilde{\Omega}_k$ as a stationary solution to Eq. (7), since it makes the integrand vanish. This *renormalized* RJ distribution is consistent with the ZM prediction [Eq. (4)], but deviates from the classical RJ distribution $\langle |a_k|^2 \rangle = \theta/|k|^\alpha$ as predicted by WT theory using the bare Hamiltonian \mathcal{H} , especially for high k and strong nonlinearity. As a verification of the validity of the effective kinetic equation (7), Fig. 3 shows that $\langle |a_k|^2 \rangle$ obtained numerically using time average from the original dynamics (2) in equilibrium agrees very well with the prediction $n_k = \theta/\tilde{\Omega}_k$.

The Kolmogorov-Zakharov nonequilibrium spectra [3] do not satisfy Eq. (7), since $\tilde{\Omega}_k$ no longer has a simple power law scaling as in the bare $\omega_k = |k|^\alpha$. For a driven-damped MMT system [13], our simulation reveals a bifurcation of the renormalized dispersion relation. In the weakly driven, damped system, the wave system is similar to the thermal equilibrium case, i.e., $n_k \sim 1/\Omega_k^R$ in the inertial range with the renormalized $\omega_k = \Omega_k^R$. For strong driving and damping, however, a new dispersion relation $\omega_k \sim |k|^\gamma$ with $\gamma \sim 0.55$ is observed for $\alpha = 1/2$. Furthermore, our numerical analysis shows that even in the driven, damped system, $|\mathcal{A}_{k_3 k_4}^{k_1 k_2}|$ matches well with the four-wave resonance structure (6) by its corresponding renormalized dispersion relation.

In conclusion, a new dynamical picture of WT emerges: The linear dispersion relation is effectively renormalized, allowing one to treat systems with strong nonlinearities.

This renormalization can create new resonances that are not present in the bare resonances, giving rise to WT dynamics which cannot be captured by the classical WT theory. Going beyond the classical *perturbative* perspective of WT, our work has revealed a *nonperturbative* nature of WT with the spectrum n_k of WT dynamics determined by an intertwining *self-consistent* process: The trivial resonant scatterings of waves off of background waves characterized by n_k control the true, renormalized, dispersion relation. This renormalized dispersion relation, in turn, controls nontrivial resonances of the full dynamics, thus giving rise to a self-consistent wave spectrum n_k .

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*cai@cims.nyu.edu.

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