Generalization of the Fractal Einstein Law Relating Conduction and Diffusion on Networks

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We settle a long-standing controversy about the exactness of the fractal Einstein and Alexander-Orbach laws by showing that the properties of a class of fractal trees violate both laws. A new formula is derived which unifies the two classical results by showing that if one holds, then so must the other, and resolves a puzzling discrepancy in the properties of Eden trees and diffusion-limited aggregates. We also conjecture that the result holds for networks which have no fractal dimension. The failure of the classical laws is attributed to anisotropic exploration of the network by a random walker. The occurrence of this newly revealed behavior means that the conventional laws must be checked if they, or numerous results which depend on them, are to be applied accurately.

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Numerous physical systems have been shown to exhibit anomalous electrical and diffusive transport properties characterized by power laws with noninteger exponents [1–12]. In particular, the electrical resistivity increases with distance r as $\rho(r) \sim r^{\zeta}$, and the distance traveled by a random walker scales with time as $\langle r \rangle \sim t^{1/d_w}$, where d_w, the random walk dimension, is generally not 2. In addition to quantifying mass and electronic transport in materials, $\langle r \rangle$ and ρ can be linked to a variety of problems such as oil recovery in porous rocks [3], chemical reaction rates [4], cellular processes [5,6], and first passage times associated with viral infections and animal foraging strategies [4].

The interrelationships between the anomalous power law exponents are the structure-property correlations for fractal networks. The Alexander and Orbach (AO) [12] law states that $\bar{d} = 2d_f/d_w$, where $\bar{d}$ and d_f are the spectral and fractal dimensions, respectively. Rammal and Toulouse [8] predicted that the electrical and diffusive properties were connected by the formula $\zeta = d_f(2 - \bar{d})/\bar{d}$. Combining both results gives the fractal Einstein (FE) law $\zeta = d_w - d_f$, so called because it can also be derived from a result due to Einstein [1]. These results continue to be employed to predict the properties of fractal networks and either underpin, or are used directly in, the study of complex networks [4,5]. Although there is a preponderance of evidence in their favor, the exactness of both formulas is controversial [9,10]: Computational results for diffusion-limited aggregates (DLAs) [13] and Eden trees [14] appear to violate the laws.

In 1989, Telcs [15] provided sufficient conditions for the FE law to hold. These include a technical "smoothness" criterion on the electrostatic potential (which must be measured to test if it holds). The method was adapted to further prove [16] that if $\zeta = d_w - d_f$, then $\bar{d} = 2d_f/d_w$ for loopless smooth networks with $\zeta > 0$; however, the precise connection between the two laws for general networks has not been established. Also note that there exist important inhomogeneous networks [17] (i.e., networks which have no d_f) which have well defined electrical and diffusive properties, for which no analog of the FE law exists.

The unexplained exceptions to the conventional laws, the absence of a proof based on network topology alone, and the lack of a structure-property law for inhomogeneous networks highlight the need for ongoing study. In this Letter, we investigate the properties of a class of fractal trees [18–21] which violate both conventional laws. The network (Fig. 1) is made by taking a base unit, doubling its size, and attaching u_i ($i = 1, 2$) copies of the rescaled unit to each of the two end points of the base. Continuing the process indefinitely gives an infinite network with $d_f = \ln[2(u_1 + u_2)]/\ln(2)$ and $d_w = 2$. To find the resistance ρ^* [18], the infinite network is represented by three resistors (Fig. 1): the stem of length 1 and two branches of resistivity ρ_1 and ρ_2. Kirchoff's law states that $\rho_1 = 1 + 2\rho^*/u_1$ and $\rho_2 = 2 + 2\rho^*/u_2$, where $2\rho^*$ is the resistance of the infinite branches connected to the two end points of the base unit. Because the shape of each branch is identical to the original infinite network (but each element is twice as long), their resistance is $2\rho^*$. Kirchoff's law for the

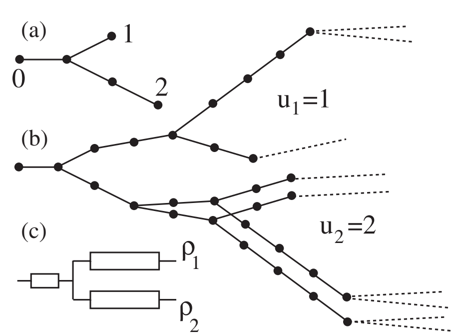

FIG. 1. A model fractal tree. (a) The base unit; (b) the second generation of the network obtained by attaching multiple copies of the rescaled base unit to the end points of the base unit; (c) an equivalent resistor network used to calculate ρ^*.

three-resistor circuit gives $\rho^* = 1 + 1/(1/\rho_1 + 1/\rho_2)$, which is quadratic in ρ^*.

Renormalization methods can be used to derive [22] the spectral dimension via the relation $c(0, t) \sim t^{-\bar{d}/2}$, where $c(0, t)$ is the probability that a random walker visits its origin at time t. For $(u_1, u_2) \neq (1, 1)$, the quadratic for ρ^* has a finite positive root, and it can be shown that

$$\bar{d} = \log_2\left(\frac{2[(2\rho^* + 3u_2)u_1 + 2\rho^* u_2]^2}{u_1 u_2(u_1 + 12\rho^* + 4u_2) + 4\rho^{*2}(u_2 + u_1)}\right).$$

For the case $(u_1, u_2) = (1, 1)$, the quadratic has no positive solutions (implying ρ^* is infinite), and we can show $\bar{d} = \ln[2(u_1 + u_2)]/\ln(2)$. The spatial dependence of the network resistance for large x is obtained from the relation

$$\rho(x) = 1 + \left\{\left[1 + \frac{2}{u_1}\rho\left(\frac{x}{2}\right)\right]^{-1} + \left[2 + \frac{2}{u_2}\rho\left(\frac{x}{2}\right)\right]^{-1}\right\}^{-1}.$$

For $\bar{d} > 2$, the resistivity must have the form $\rho(x) = a_0 + a_1 x^{\zeta}$ ($a_0 = \rho^*$) (see [9], p. 468). Taking series expansions leads to $\zeta = 2 - \bar{d}$. The same result is achieved if $\bar{d} < 2$.

The properties of the network violate the AO and FE laws unless $u_1 = u_2 = 1$ or $u_1 = 2u_2$. For example, $u_1 = 1$ and $u_2 = 20$ give $\bar{d} \approx 4.37$, which does not equal $2d_f/d_w = 5.39$. Similarly, $\zeta = 2 - \bar{d} = -2.37$ disagrees with the prediction $\zeta = d_w - d_f = -3.34$. Standard computations were used [22] to verify $\bar{d}$, d_w, and d_f.

In order to derive a new relationship between the electrostatic and diffusive properties of a network (similar ideas are used in Refs. [4,9,15,18]), consider the concentration field generated by the release of a random walker at the origin at every time step. This concentration is exactly given by $C(r, t) = \int_0^t c(r, \tau) d\tau$, where $c(r, t)$ is the probability of finding a random walker at r, after time t, if a single walker is released at the origin at $t = 0$. To link the dynamic and static problems, the integration is terminated at $T = (R/b)^{d_w}$, where b is a number of order 1. As $R = bT^{1/d_w}$ is a typical distance reached by the initial walker after time T, only a very small proportion of the $T + 1$ walkers released will exceed this radius; hence $C(r, T) \approx 0$ for $r \geq R$. In the central region the spatial concentration profile $C(r, T)$ is assumed to have equilibrated and therefore satisfies Laplace's equation. The boundary conditions correspond to the potential $\phi(r; R)$ on a finite network grounded at radius R due to the supply of unit current ($I = 1$) at the origin. Therefore the resistance is $\rho(R) = \phi(0; R)/I \approx C(0, (R/b)^{d_w})/I$ giving

$$\rho(R) \approx \int_0^{(R/b)^{d_w}} c(0, t) dt \sim \begin{cases} R^{d_w(2-\bar{d})/2} & \bar{d} < 2, \\ \log(R) & \bar{d} = 2, \\ \rho^* - QR^{d_w(2-\bar{d})/2} & \bar{d} > 2, \end{cases}$$

where Q is a constant. This exactly matches the known scaling behavior of the resistance if

$$\zeta = d_w(2 - \bar{d})/2. \quad (1)$$

Note that the spectral dimension [23], and hence ζ, are site independent, even though ρ^* can vary from site to site if $\bar{d} > 2$. Equation (1) is exact for the fractal trees depicted in Fig. 1 as $d_w = 2$. Computations of $C(0, (R/b)^{d_w})$ (obtained using the diffusion equation) and $\rho(R)$ (Laplace's equation) shown in Fig. 2 support the validity of the approximation $\rho(R) \approx C(0, (R/b)^{d_w})$. The resultant estimates of ζ are in good agreement with the analytic result $\zeta = 2 - \bar{d}$. To rule out the possibility that Eq. (1) is restricted to $d_w = 2$, we have also confirmed [22] the result for trees where the branches are deterministic trees of increasing iteration.

Table I shows available data [10] for Eden trees [14,24] and DLA clusters [11,13]. The resistance of loopless fractals is proportional to the length of the shortest path ℓ between two sites which scales [10] as $\ell \sim r^{d_{\min}}$ (so $\zeta = d_{\min}$). Equation (1) is seen to provide a good estimate of ζ for DLA and Eden trees in three dimensions (in contrast to the FE law). In two dimensions, Eq. (1) is superior to the FE law for Eden trees, whereas for DLA both Eq. (1) and the FE law have a similar level of accuracy and are consistent with $\zeta = 1$. Data for Eden trees were obtained for relatively small clusters, and it would be useful to reconsider the calculations.

To understand why the AO and FE laws can fail, it is necessary to consider the assumptions underlying their derivation. In general, the probability that a walker released at $\mathbf{r} = 0$ at $t = 0$ will be at the point $\mathbf{r}$ on a network after time t will depend on the direction of $\mathbf{r}$ as well as its magnitude $r = |\mathbf{r}|$. This probability is denoted as $c_a(\mathbf{r}, t)$, where the subscript a (anisotropic) differentiates it from

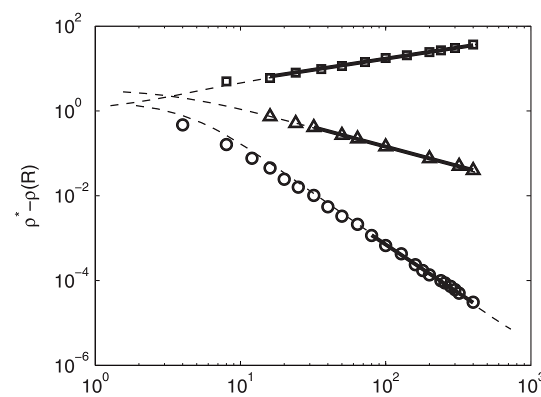

FIG. 2. The computed resistivity $\rho(R)$ of three networks (symbols) alongside the concentration $C(0, (R/b)^{d_w})$ (dashed lines). The solid lines represent lines of best fit to $\rho(R)$. The network with $(u_1, u_2) = (1, 20)$ (○), $\zeta = -2.37$ (best fit: -2.28); the network with $(u_1, u_2) = (1, 3)$ (△), $\zeta = -0.941$ (best fit: -0.92); a network with $\zeta = 0.5$ (□) obtained by taking $(u_1, u_2) = (1, 1)$ and quadrupling the branch lengths at each iteration [22] (best fit: 0.527). Here b (chosen by eye) shifts the dashed lines horizontally ($b = 0.9$, 1.1, and 1.1, respectively).

TABLE I. Equation (1) provides a significantly better estimate of ζ than the fractal Einstein relationship for Eden trees and three-dimensional DLA clusters. The table is adapted from Ref. [10] using data from Refs. [11,13,14,24].

Fractal	$\bar{d}$	d_w	d_f	ζ—meas.	ζ—FE law	ζ—Eq. (1)
DLA 2D	1.20 ± 0.05 [13]	2.64 ± 0.05 [13]	1.70 ± 0.02 [13]	1.00 ± 0.02 [11]	0.94 ± 0.07	1.05 ± 0.09
DLA 3D	1.35 ± 0.05 [13]	3.19 ± 0.08 [13]	2.48 ± 0.02 [13]	1.02 ± 0.03 [11]	0.71 ± 0.10	1.04 ± 0.10
Eden tree 2D	1.22 ± 0.02 [14]	2.82 ± 0.06 [24]	2	1.22 ± 0.02 [14]	0.82 ± 0.07	1.10 ± 0.05
Eden tree 3D	1.32 ± 0.02 [14]	3.85 ± 0.15 [24]	3	1.32 ± 0.02 [14]	0.85 ± 0.07	1.31 ± 0.09

the function $c(r, t)$ used above. The two functions can be related by $c(r, t) = S^{-1} \int_S c_a(\mathbf{r}, t) dS$, where $S(r) \sim r^{d_f - 1}$ is the area (mass) of the fractal at radius r. $c(r, t)$ can be regarded as a *network-spherical* average, because the average on the shell is taken only over the regions occupied by the network. Equivalently, it can be called [25] the average probability *per site*.

After time t, a walker released from the origin will on average have explored a region of radius $R \sim t^{1/d_w}$ of volume $V(R) \sim R^{d_f}$, so $\int_{V(R)} c_a(\mathbf{r}, t) dV \approx 1$. It is generally assumed that $V(R)$ is explored approximately uniformly [10], so $c_a(0, t) \int_{V(R)} dV \approx 1$ or $c(0, t) \approx 1/V(R(t)) \sim t^{-d_f/d_w}$. This provides the rationale behind the AO law. In particular, this requires that the concentration field is approximately isotropic *on the network* [$c_a(\mathbf{r}, t) \approx c(r, t)$]. If this is not true, the volume explored by the walker will generally not be $V(R(t))$. Data shown in Fig. 3 confirm that $c_a(\mathbf{r}, t) \approx c(r, t)$ for the fractal tree with $(u_1, u_2) = (2, 1)$. This tree's properties follow the AO law exactly. In contrast, $c_a(\mathbf{r}, t)$ is seen to be strongly anisotropic for the fractal tree with $(u_1, u_2) = (1, 20)$. We attribute the breakdown of the AO law to this nonuniform exploration of the network.

A similar requirement of uniformity is implicitly assumed in the derivation of the FE law. This is clearly seen in an examination of the total current flow I through a shell of thickness Δr. By definition $I = ne \times dS \times \Delta r/(\Delta t)$, where n is the charge carrier density, e is the carrier charge, dS is an element of area, and Δt is the time it takes a charge to cross the shell. Now the time scale for diffusing a distance r is $t \sim r^{d_w}$, so $\Delta t \sim r^{d_w - 1} \Delta r$. Summing over the shell gives

$$I \sim \Delta r \int_S \frac{dS}{\Delta t} \approx \Delta r \frac{S}{\Delta t} \sim r^{d_f - d_w},$$

and therefore $\rho = V/I \sim r^{d_w - d_f}$, which reproduces the FE law. Although the argument assumes Δt is independent of direction, this is only strictly true if $c_a(\mathbf{r}, t)$ is uniform over the shell. Therefore, if diffusion exhibits preferential directions on the network [as it does for the tree with $(u_1, u_2) = (1, 20)$], the FE law may be invalid.

The fractal tree provides a concrete example of the qualitative balance arguments expressed above. Recall that the FE law is obeyed when $u_1 = 2u_2$. Rearranging the expressions for ρ_1 and ρ_2 gives $\rho_2 u_2 - \rho_1 u_1 = 2u_2 - u_1$; hence the condition $u_1 = 2u_2$ implies $\rho_2/\rho_1 = I_1/I_2 = u_1/u_2$, where I_i $(i = 1, 2)$ are the currents on each branch. As the ratio of the masses of the branches is u_1/u_2, it is seen that conventional scaling holds because the mass and current on different branches extending from a node are balanced. However, for the case $(u_1, u_2) = (1, 20)$, a significant mass-current imbalance occurs; although there is 20 times more mass in branch 1 than branch 2, the current is only about $I_1/I_2 \approx 3$ times greater.

It is interesting to consider why Eq. (1) holds for anisotropic $c_a(\mathbf{r}, t)$. It is obviously possible to define $\langle r(t) \rangle$ (and hence d_w) by a suitable average. It is also clear that resistance (and hence ζ) can be defined by the current induced by a unit potential applied between an origin and an earthed shell at $|\mathbf{r}| = R$. The field $\phi_a(\mathbf{r}; R)$ is equipotential on this shell, but clearly its approximation $\int_0^{(R/b)^{d_w}} c_a(\mathbf{r}, t) dt$ is not. Recall, however, that the approximation needs only to be accurate at the origin, and by implication this is so. As d_w and ζ both depend on spherical averages of $c_a(\mathbf{r}, t)$, it is not unexpected that their interrelationship [Eq. (1)] does not independently incorporate the mass via d_f.

As the derivation of Eq. (1) does not require the network to have a fractal dimension, we propose that Eq. (1) holds for inhomogeneous networks [17]. Although d_w only technically exists for fractal networks, an analogous exponent α is defined by $\langle r^2 \rangle \sim t^\alpha$, whereby Eq. (1) becomes $\zeta = (2 - \bar{d})/\alpha$. Consider the fractal tree, which becomes in-

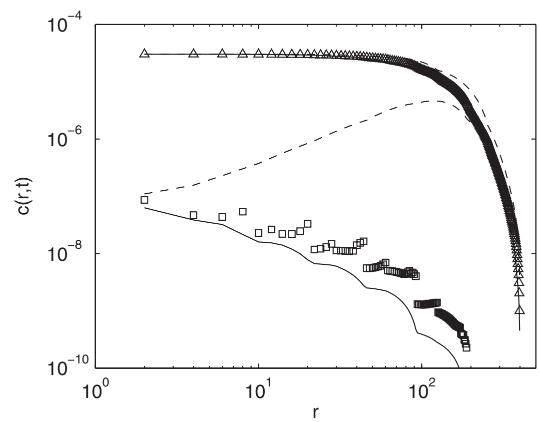

FIG. 3. The spherically averaged functions $c(r, t)$ for the fractal tree with $(u_1, u_2) = (2, 1)$ (△) and $(u_1, u_2) = (1, 20)$ (□). The lines indicate the highest and lowest values of $c_a(\mathbf{r}, t)$ at each r.

homogeneous as $u_2 \to \infty$ ($d_f \to \infty$). The argument used to show $\zeta = 2 - \bar{d}$ is not altered in this limit, so Eq. (1) continues to hold. A second example is provided by the comb lattice [1] with an infinite spine and infinite teeth. This inhomogeneous network has $\bar{d} = 3/2$ and $\alpha = 1$ (since the teeth are one-dimensional). The exponent ζ can be easily found if the teeth are folded against the spine (this will not affect $\bar{d}$ and α). If the lattice is earthed along a line perpendicular to the spine (at node 0), the resistance between the nth node to the left of the line and the line is $\rho_n = 1/(1/n + 1/\rho_{n-1})$, with $\rho_1 = 1/2$. For large n, $\rho_n \approx \rho_{n-1}$, and the solution of the quadratic is $\rho_n = \sqrt{n}$, so $\zeta = 1/2$, which is consistent with $\zeta = (2 - \bar{d})/\alpha$.

Our main result [Eq. (1)] generalizes the FE law to fractals which to not adhere to the AO law as well as to networks that do not have a well defined fractal dimension. This is practically useful as ζ is harder to compute than $\bar{d}$ and d_w [10]. Additionally, Eq. (1) represents a conceptual advance by establishing a simple and direct unification of the Alexander-Orbach and fractal Einstein laws; if one holds, then so does the other. For example, Eq. (1) shows that $\bar{d} = 2d_f/d_w$ if, and only if, $\zeta = d_w - d_f$, irrespective of the presence of loops or the sign of ζ (cf. [16]).

We have attributed the breakdown of the FE law to the existence of anisotropic probability and electrostatic fields on the network. This corresponds to the nonuniform exploration of sites at a given distance from the origin by a random walker. From this perspective, transport is *defined* to be isotropic *on the network* if the AO and FE laws hold. Note that this *a posteriori* definition of isotropy skirts the interesting problem of bounding the degree of allowable local anisotropic variation in the probability and potential [15] fields. It is simple to visualize highly anisotropic fields on DLA clusters: A tiny fraction of the total mass will carry the entire current, while long massive branches that emerge near the origin and fail to touch the boundary sphere will be equipotential [so $\phi(r)$ will differ greatly from $\phi_a(\mathbf{r})$ in many places]. Further studies of the tree model show that the AO and FE laws can still fail when the branches contain internal loops of increasing scale. A different model is needed to determine if anisotropic exploration can occur if the branches have some degree of interconnection.

The success of the AO and FE laws for the percolation model and deterministic fractals (such as the Sierpinski triangle), coupled with the absence of conclusive counterexamples, has led to their wide acceptance. For example, the FE law [5] and recent source-to-target times [4] (derived using a propagator which relies on the AO law) have recently been proposed to model the properties of complex networks which are defined by highly variable degree distributions and exhibit modularity. Aspects of these characteristics are represented in the trees considered here: The branches communicate with the wider network through a small number of links, and the forks have strongly asymmetric coordination numbers ($u_1 + 1 = 2$ and $u_2 + 1 = 21$). It is therefore generally important to test the assumption of isotropic transport (via the AO or FE law).

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