Identifying Phases of Quantum Many-Body Systems That Are Universal for Quantum Computation

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Quantum computation can proceed solely through single-qubit measurements on an appropriate quantum state, such as the ground state of an interacting many-body system. We investigate a simple spin-lattice system based on the cluster-state model, and by using nonlocal correlation functions that quantify the fidelity of quantum gates performed between distant qubits, we demonstrate that it possesses a quantum (zero-temperature) phase transition between a disordered phase and an ordered "cluster phase" in which it is possible to perform a universal set of quantum gates.

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Measurement-based quantum computation (MBQC) is a fundamentally new approach to quantum computing. MBQC proceeds by using only local adaptive measurements on single qubits. No entangling operations are required; all entanglement for the computation is supplied by a fixed initial resource state on a lattice of qubits. The canonical example of such a resource state is the so-called *cluster state* [1,2]. Although a handful of other universal resources have recently been identified [3–6], there currently exists very little understanding of precisely which properties of quantum states allow for universal MBQC. For example, given a state that is slightly perturbed from the cluster state, it is not currently known how to determine if it is a universal resource. New theoretical tools are required to identify the properties of potential resource states that allow for universal MBQC.

A useful perspective to approach this problem is to view the resource state for MBQC as the ground state of a strongly coupled quantum many-body system. With this perspective, we propose that the ability to perform MBQC is a type of quantum order—one which can be identified using appropriate correlation functions as order parameters. We show that a natural choice for such correlation functions are the expectation values of nonlocal strings of operators that can be identified with measurement sequences for performing quantum logic gates within MBQC. One way of understanding MBQC is that, by means of a set of local measurements, it is possible to prepare the resource states required for gate teleportation [2,7,8] between distant components of the many-body system. The performance of the MBQC scheme can be characterized by calculating the fidelity of the prepared resource state with the ideal one [9]. This fidelity will depend on a set of nonlocal correlation functions as a result of the many local measurements that are required to prepare the resource state. (Because the fidelity of the identity gate is quantified by the ability to prepare an entangled state between two distant qubits using local measurements, it is closely related to the much-studied property of *localizable entanglement* [10].) We show that, for the cluster-state implementation of MBQC, the specific correlation functions corresponding to any gate can be calculated, and we investigate a specific model where the fidelities of a gate set indeed serve as order parameters identifying a *cluster phase*. This result suggests the existence of spin systems that possess a phase for which any state is a universal resource for MBQC. These methods provide new tools for identifying properties of quantum many-body systems that are required for MBQC.

Consider the following model system. The cluster state on a lattice $\mathcal{L}$ is defined as the unique $+1$ eigenstate of a set of stabilizer operators $K_\mu = X_\mu \prod_{\nu\sim\mu} Z_\nu$, where X_μ (Z_μ) is the Pauli X (Z) operator at site μ and where $\nu \sim \mu$ denotes that ν is connected to μ by a bond in the lattice $\mathcal{L}$. The Hamiltonian $H = -\sum_{\mu\in\mathcal{L}} K_\mu$ has the cluster state as its unique ground state [11]. Although the terms in this Hamiltonian are many-body interactions, it can be realized as the effective low-energy theory of a Hamiltonian consisting only of two-body terms [12].

As a model system to consider how robust is this Hamiltonian in the presence of local perturbations, we supplement it with a local field term,

$$H(B) = -\sum_{\mu\in\mathcal{L}} (K_\mu + BX_\mu), \tag{1}$$

representing a local transverse field with magnitude B. We refer to a lattice with this Hamiltonian as the *transverse-field cluster model* (TFCM), and we will demonstrate the existence of a single zero-temperature phase transition in the ground state of such models on both a 1D line and a 2D square lattice, separating a disordered phase from a "cluster phase." Rather than solving these models explicitly, we explore duality transformations that relate these models to others with well-understood phases and order parameters. We then demonstrate that the order parameters of these models, mapped back to the TFCM, are precisely equivalent to the correlation functions in the cluster state that quantify the fidelity of the identity gate (i.e., teleportation) in MBQC. That is, the ability to perform the identity gate over a long range serves as an order parameter for this phase; similar results hold for other single-qubit gates as well. In addition, in two dimensions, we perform a similar analysis of the two-qubit controlled SIGN (CSIGN) gate, $\exp(i\pi|1\rangle\langle 1| \otimes |1\rangle\langle 1|)$, which together with our single-qubit gates yields a universal gate set for MBQC. (In contrast, the case with a local longitudinal field instead of a transverse one was investigated in [13]; this model demonstrates no such phase but nevertheless can still allow for MBQC for some range of parameters).

General properties of the transverse-field cluster model.—We first present some general properties of the TFCM that are valid in any dimension and on many lattices, before investigating one- and two-dimensional models in detail. An immediate observation is that this model is *self-dual*. The canonical transformation of Pauli operators given by applying the CSIGN operation between all neighboring pairs of qubits takes $K_\mu \leftrightarrow X_\mu$, and thus the Hamiltonian (1) transforms as $H(B) \to BH(1/B)$. This self-duality ensures that, if this model has a single quantum phase transition in the range $B > 0$, then it must occur at $B = 1$.

Also, consider lattices which are bipartite, meaning we can divide the sites into two subsets $\mathcal{L}_r$ and $\mathcal{L}_b$, labeled red and blue, such that the neighbors of any site are all of the other color. With this coloring, the Hamiltonian (1) can be written as the sum of two commuting terms, $H = H_r + H_b$, where

$$H_r = -\sum_{\mu\in\mathcal{L}_b} K_\mu - B\sum_{\mu\in\mathcal{L}_r} X_\mu, \tag{2}$$

with H_b consisting of the remaining terms. In the following, we present mappings of H_r (equivalently, H_b) in one and two dimensions to known models, which allows us to identify the phases and relevant order parameters.

One dimension.—Consider the TFCM on a 1D lattice with fixed boundary conditions—a line. A state of a 1D lattice cannot serve as a universal resource for MBQC; however, it will be illustrative to consider this model as a prelude for studying higher dimensions. The Hamiltonian (1) on a line with boundary terms is

$$H(B) = -\sum_{i=2}^{N-1}(Z_{i-1}X_iZ_{i+1} + BX_i) - X_1Z_2 - BX_1 - Z_{N-1}X_N - BX_N. \tag{3}$$

The ground state of this Hamiltonian is nondegenerate, and for $B = 0$ is given by the 1D cluster state on a line. Pachos and Plenio [14] have shown explicitly that this model (with periodic boundary conditions) exhibits a quantum phase transition at $|B| = 1$, and that the localizable entanglement length remains infinite for all values $|B| < 1$. Their method makes use of the Jordan-Wigner transformation to yield a linear fermionic system. We provide a more direct transformation to a known model—the transverse-field Ising model [15]—which provides a natural generalization to higher-dimensional lattices.

Our duality transformation is as follows. On red (even) sites, the Pauli operators transform as

$$X_{2j} \to \bar{X}_{2j}, \qquad Z_{2j} \to \Big(\prod_{k=1}^{j}\bar{X}_{2k-1}\Big)\bar{Z}_{2j}. \tag{4}$$

On blue (odd) sites, the Pauli operators transform as

$$X_{2j-1} \to \bar{X}_{2j-1}, \qquad Z_{2j-1} \to \bar{Z}_{2j-1}\Big(\prod_{k=j}^{N}\bar{X}_{2k}\Big). \tag{5}$$

This mapping is canonical, meaning the new Pauli matrices $\bar{X}_j$ and $\bar{Z}_j$ satisfy the correct commutation and anticommutation relations. An illustration of this transformation is presented in Fig. 1(a). We emphasize that this duality transformation is nonlocal, and thus the properties of a system for MBQC are not preserved under this mapping. However, as we now demonstrate, the phases and order parameters of this dual model are well-studied and will allow us to completely classify the phases as well as calculate the fidelities of the MBQC quantum gates in the original TFCM.

We consider only the case where N is even. In terms of transformed Pauli operators, the Hamiltonian H_r acts only on red sites and has the form

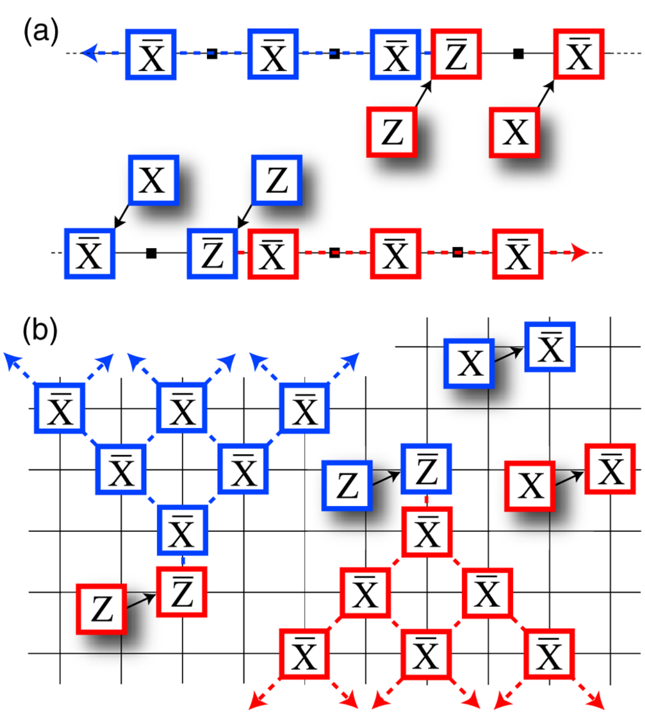

FIG. 1 (color online). (a) The duality transformation of Eqs. (4) and (5) on a 1D line. (b) A generalization of this duality transformation to a 2D square lattice.

$$H_r = -\bar{Z}_2 - \sum_{i=2}^{N/2} (\bar{Z}_{2(i-1)}\bar{Z}_{2i} + B\bar{X}_{2i}). \qquad (6)$$

The Hamiltonian H_b is similar, acting only on blue sites, with a $\bar{Z}$ boundary term at $j = N$. This mapping on the TFCM, then, yields two identical transverse-field Ising models, one on each of $\mathcal{L}_r$ and $\mathcal{L}_b$. Each has a local $\bar{Z}$ field term which breaks the symmetry in the ordered ($|B| < 1$) phase and specifies a unique ground state. The ground state of the total lattice is then nondegenerate and is given by the product state of these two unique ground states.

The solution to this known model allows us, via the duality transformation, to completely characterize the TFCM. For example, the phases of the TFCM are specified by the well-studied phases of the transverse-field Ising model; in particular, there is a unique quantum phase transition at $|B| = 1$ [16]. Also, the well-known order parameters for the transverse-field Ising model can be mapped, using the duality transformation, to order parameters for the TFCM. In the ordered phase of the transverse-field Ising model, the correlation functions $\langle \bar{Z}\bar{Z}\rangle$ (for both colors) are long ranged. [Specifically, $\lim_{k\to\infty}\langle \bar{Z}_i\bar{Z}_{i+k}\rangle = (1 - |B|^2)^{1/4}$ for $|B| < 1$ [16].] We can use this result to make a corresponding statement about correlation functions for the TFCM. By reversing the duality transformation, we have

$$\langle \bar{Z}_{2i-1}\bar{Z}_{2j-1}\rangle \rightarrow \left\langle Z_{2i-1}\left(\prod_{k=i}^{j-1} X_{2k}\right)Z_{2j-1}\right\rangle = \left\langle \prod_{k=i}^{j-1} K_{2k}\right\rangle, \qquad (7)$$

$$\langle \bar{Z}_{2i}\bar{Z}_{2j}\rangle \rightarrow \left\langle Z_{2i}\left(\prod_{k=i}^{j-1} X_{2k+1}\right)Z_{2j}\right\rangle = \left\langle \prod_{k=i}^{j-1} K_{2k+1}\right\rangle. \qquad (8)$$

That is, in the phase $|B| < 1$ wherein $\langle \bar{Z}\bar{Z}\rangle$ is long-ranged, the stringlike operators corresponding to the product of even (or odd) stabilizers K_i in the TFCM are also long-ranged, with the limiting value $(1 - |B|^2)^{1/4}$. These two correlation functions are all that is needed to calculate the fidelity of the resource state for the identity gate with the ideal maximally-entangled state, and it is found to be $>1/4$ for all $|B| < 1$. (The average fidelity of a randomly chosen state yields $1/4$.) The same calculation for other single-qubit Clifford gates [17] and for an arbitrary Z rotation $U_z(\theta) = \exp(-i\theta Z)$ (a non-Clifford gate), yields the same result [9].

Thus, this duality transformation has allowed us to prove our desired results: First, that the TFCM does indeed possess a phase, given by $|B| < 1$, which we denote the *cluster phase*. The order parameters of this phase, given by products of even or odd stabilizer operators K_i, demonstrate that quantum gates can be performed with high fidelity (relative to a randomly chosen state) using any state within this phase. The ground states in this phase are indeed "robust" against variations in the precise value of B. However, the one-dimensional cluster state is not a universal resource for MBQC, and so we direct our attention to a two-dimensional model.

Two dimensions.—We consider a square lattice; the cluster state on this lattice is a universal resource for MBQC. This lattice is bipartite, and thus we can define the commuting Hamiltonians H_r and H_b as above. We use a natural generalization of the 1D duality transformation, as follows. On red sites, Pauli operators transform as $X_\mu \rightarrow \bar{X}_\mu$ and $Z_\mu \rightarrow (\prod_{\mu'>\mu}\bar{X}_{\mu'})\bar{Z}_\mu$, whereas on blue sites, $X_\mu \rightarrow \bar{X}_\mu$ and $Z_\mu \rightarrow \bar{Z}_\mu(\prod_{\mu'<\mu}\bar{X}_{\mu'})$. Here, $\mu' > \mu$ ($\mu' < \mu$) denotes that μ' lies in the upper (lower) cone relative to μ as in Fig. 1(b). Again, one can easily verify that this transformation is canonical.

Under this mapping, each stabilizer maps to a monochromatic operator consisting only of $\bar{Z}$ terms. Nonboundary stabilizers map to products of four $\bar{Z}$ operators on the corners of a fundamental plaquette $\square$; boundary conditions can be chosen such that boundary stabilizers map to two-$\bar{Z}$ and one-$\bar{Z}$ terms. The Hamiltonian H_r (H_b) on $\mathcal{L}_r$ ($\mathcal{L}_b$) maps to

$$H = -\sum_{\square} \begin{matrix}\bar{Z} & \bar{Z}\\ \bar{Z} & \bar{Z}\end{matrix} - B\sum_{\mu}\bar{X}_\mu, \qquad (9)$$

plus boundary terms (not shown) which ensure a nondegenerate ground state for all B. This model possesses a phase transition at $|B| = 1$ [18,19]. Thus, through this duality map, we know that the 2D TFCM has a phase transition at $|B| = 1$, and we use the term *cluster phase* to denote the $|B| < 1$ phase. In addition, this model of Eq. (9) is dual to the *anisotropic quantum orbital compass model*

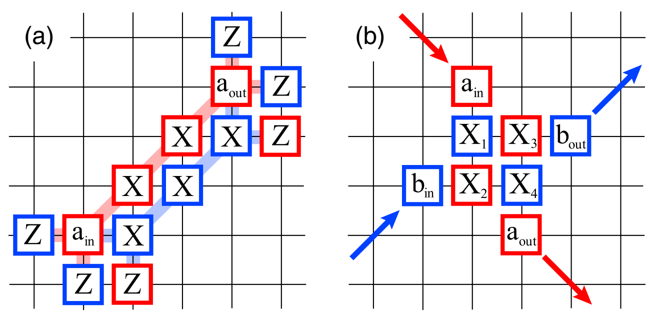

FIG. 2 (color online). (a) A measurement pattern on the cluster state that localizes entanglement between sites $a_{\rm in}$ and $a_{\rm out}$, where X (Z) denotes a measurement in the X basis (Z basis). The two stringlike stabilizers, centered on sites connected by the shaded red and shaded blue diagonal lines, have long-ranged expectation values in the $|B| < 1$ phase; these correlation functions directly quantify the fidelities of single-qubit gates between $a_{\rm in}$ and $a_{\rm out}$ in MBQC. (b) The measurement sequence corresponding to the CSIGN gate between a and b. The expectation of four stabilizers characterizes the CSIGN gate: $K_{a_{\rm in}}K_3K_{a_{\rm out}}$, $K_{b_{\rm in}}K_4K_{b_{\rm out}}$, K_1K_4, and K_2K_3. These stabilizers can be appended with diagonal strings of red (blue) stabilizers in the direction of the arrows [and terminated with Z measurements as in (a)] to reach distant qubits. With X measurements on qubits 1–4, the resulting state provides the CSIGN transformation.

(AQOCM) [20–23], with a mapping that also locally maps the boundary terms. The key advantage of the AQOCM is that it contains only two-body terms in the Hamiltonian, and is therefore very amenable to numerical investigation. For example, the projected entangled-pair state algorithm applied to this model provides very strong evidence that the phase transition is first order [24]. The model also possesses correlation functions for an Ising order parameter that simulations indicate are long-ranged for $|B| < 1$ [24].

Inverting this duality transformation and returning to the TFCM, these Ising-type correlation functions map onto strings of monochromatic stabilizers along diagonal lines in the square lattice [see Fig. 2(a)]. Again using the correlation functions for single-qubit gates given in [9], we find that these strings of monochromatic stabilizers characterize the fidelities of the identity gate and a generating set of single-qubit gates between two distant points, and serve as order parameters for the cluster phase.

In addition, in this 2D model, we can consider two-qubit gates. We make use of the elementary measurement pattern for a CSIGN gate on two qubits which are subsequently swapped, as given in Refs. [2,9] and shown in Fig. 2(b). The desired long-ranged correlation functions on the AQOCM are of the form of 4-body correlators $\langle \tilde{Z}_{(i,j_0)} \tilde{Z}_{(i,j_*)} \tilde{Z}_{(i+1,j_*)} \tilde{Z}_{(i+1,j_1)} \rangle$, where j_* is an intermediate column between j_0 and j_1. Such 4-body correlation functions should be possible to numerically evaluate in the AQOCM using recent techniques. The CSIGN together with the above single-qubit gates yields a universal gate set, and thus the cluster phase is indeed characterized by the fidelities of a universal gate set for MBQC.

Discussion.—Using the TFCM as an example, we have demonstrated the utility of correlation functions corresponding to quantum gates as order parameters to identify a phase according to its usefulness for MBQC. The perspective of quantum-computational universality of a state as a new type of quantum order may assist in identifying new quantum systems that can be used for MBQC.

The behavior of the TFCM contrasts with the model considered in [13], which is the cluster-state Hamiltonian perturbed by a local Z field. In that model, the gate correlation functions discussed in this Letter become short-ranged at any nonzero perturbation. However, by preprocessing with certain local filtering operations, it is still possible to perform MBQC for sufficiently low field and sufficiently low temperature [13]. Unlike the TFCM, the model of [13] does not undergo a phase transition. These behaviors are very reminiscent of the quantum Ising model in one dimension: where there is a broken symmetry that disappears at a phase transition for sufficiently large transverse field but longitudinal fields destroy the ground state order without any phase transition.

One could also ask whether these ordered phases persist to finite temperature. As our model is gapped except at the phase transition, it is possible for a finite-sized thermal system to be cooled to have arbitrarily high overlap with the ground state (although this becomes a challenge close to the phase transition). In one dimension, the fact that the transverse-field Ising model does not maintain an ordered phase at any finite temperature demonstrates that the 1D TFCM does not either. In two dimensions, it is less clear. For this reason, it would be worth investigating the TFCM on a three-dimensional lattice such as in [11], for which the $B = 0$ model is known to allow for fault-tolerant MBQC at finite temperature [11,13,25].

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