Dynamic Opinion Model and Invasion Percolation

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We propose a ''nonconsensus'' opinion model that allows for stable coexistence of two opinions by forming clusters of agents holding the same opinion. We study this nonconsensus model on lattices, several model complex networks, and a real-life social network. We find that the model displays a phase transition behavior characterized by a large spanning cluster of nodes holding the same opinion appearing when the concentration of nodes holding the same opinion (even minority) is above a certain threshold. Because of the clustering (community support) of agents holding the same opinion, these clusters cannot be invaded by the other opinion (similar to incompressible fluids). Our extensive simulations show that the nonconsensus opinion model appears to belong to the same universality class as invasion percolation.

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Social dynamics has been studied extensively in recent years using concepts and methods based on ideas from statistical physics. An important approach is complex networks, where the nodes represent agents and the links represent the interactions between them. There is considerable current interest in the problem of how two competing opinions evolve in populations [1]. Various versions of the opinion model have been proposed [1], among which are the Sznajd model [2], the voter model [3], the majority rule model [4], and the social impact model [5]. Models incorporating the evolution of two competing states can be mapped into spin models and can be applied in a much broader range of disciplines from chemistry, physics, and biology to social science [1].

The main models which are based on spin systems with short range interactions lead to a steady state with either consensus of a single opinion or equal concentrations of the two opinions [3,4]. In real life, however, a stable coexistence with unequal concentrations of two opinions is commonly seen.

In this Letter, we propose a spin-type nonconsensus opinion (NCO) model, which demonstrates novel nontrivial stable states in which stable coexistence of minority and majority opinions occurs. This stable state is reached from a random initial configuration after a dynamical process in a relatively short time. Further, we find that, when the population of one opinion is above a certain critical threshold, even still minority, a large spanning cluster of a size proportional to the total population forms. Using extensive simulations, we find that the phase transition in the NCO model belongs to the same universality class as invasion percolation with trapping (TIP) [6,7]. Once agents holding the same opinion form a cluster, where each member of the cluster gets enough community support to hold its opinion, the cluster becomes stable and cannot be penetrated by the other opinion such as incompressible fluids in TIP. This is the first time that an opinion model as well as clustering behavior in opinion formation can be mapped to a known physics percolation problem.

The structure of clusters formed by agents holding the same opinion is relevant in many real-life scenarios, such as the propagation of ideas in human populations and communications among people holding the same opinion. Thus, it is of interest to identify and understand the topological properties of the formed clusters, as well as the distribution of the sizes of clusters and the average distance between agents belonging to the same cluster.

The basic assumption of the NCO model is that opinion formation is a process where an agent's opinion is influenced both by his own current opinion and that of his friends represented as nearest neighbors in the network. This assumption implies that a person is influenced by the majority opinion of the group which includes his friends and himself. The idea of incorporating the role played by the current state of an agent on its future state has been considered elsewhere [8]. The two opinions are denoted by σ_- and σ_+. At time $t = 0$, opinions are randomly assigned to all nodes: With probability f a node will be assigned opinion σ_- and with probability $1 - f$ opinion σ_+. For a randomly chosen node i, the node and its nearest neighbors form a set of nodes A_i. At each time step, node i will convert to its opposite opinion, if it is in the local minority opinion. If the two opinions are equally represented in A_i, i will have probability p to convert to its opposite opinion and probability $1 - p$ to remain unchanged. For simplicity, we present here results only for $p = 0$ [9]. At each simulation step, every node is tested to see whether its opinion needs to be changed. All of the updates are made simultaneously at each simulation step. The system is considered to reach a stable state if no more changes occur.

The NCO model shares features with the majority-voter model [3]. The difference is that the majority-voter model, which converges to consensus in its stable state, takes into account only the opinion of neighbors of a selected node i. Here we include the opinion of node i itself into consid-

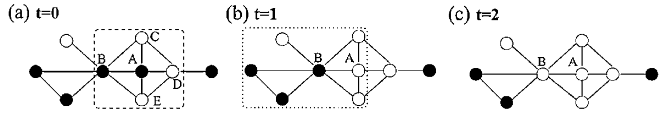

FIG. 1. Dynamics of the NCO model showing the approach to a stable state on a network with $N = 9$ nodes. For simplicity, we assume $p = 0$. (a) At $t = 0$, five nodes are randomly assigned to be σ_+ (filled circle). The remaining four nodes are assigned σ_- (open circle). In the set comprising of node A and its 4 neighboring nodes (dashed box), node A is in a local minority opinion (2 σ_+ nodes and 3 σ_- nodes), while the remaining nodes are not. At the end of simulation step $t = 0$, node A is converted into σ_-. (b) At $t = 1$, in the set of nodes comprising node B and its 6 neighboring nodes (dotted box), node B becomes in a local minority opinion (3 σ_+ nodes and 4 σ_- nodes), while the remaining nodes are not. Node B is converted into σ_- at the end of simulation step $t = 1$. (c) At $t = 2$, the nine-nodes system reaches a stable state.

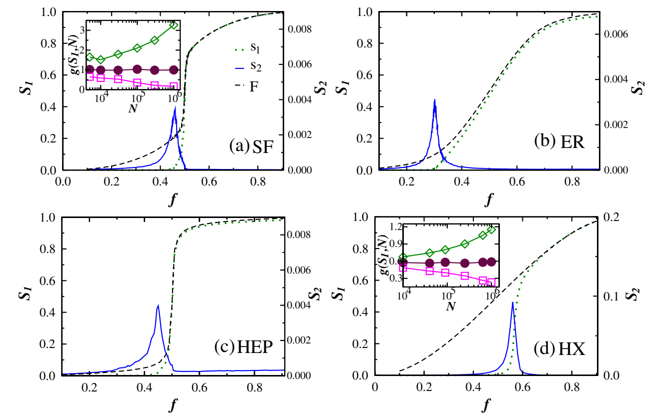

FIG. 2 (color online). Plot of the normalized size of the largest cluster s_1 (dotted line), the second largest cluster s_2 (full line), and the fraction of σ_- nodes F (dashed line) in the stable state as a function of f for (a) a SF network with $\lambda = 2.5$ and $N = 10^5$, (b) an ER network with $\langle k \rangle = 4$ and $N = 10^5$, (c) a HEP network with $\lambda \approx 2.9$, and (d) a HX lattice of size 1000×1000 [20]. Each curve represents an average over 100 realizations. The sharp increase of s_1 and the peak of s_2 at f^* indicate a second-order phase transition. The insets in (a) and (d) show $g(S_1, N)$ [Eq. (1)] as a function of N at the critical threshold f^* (●), $f^* + \Delta$ (◇), and $f^* - \Delta$ (□), where $\Delta = 0.01$ for SF and $\Delta = 0.005$ for HX. At f^*, $g(S_1, N)$ approaches a constant, indicating the size of the largest cluster S_1 proportional to N^ζ at f^* [Eq. (1)], which is another characteristic of a second-order phase transition.

eration, which makes the formation of stable clusters possible even for nonzero p.

A demonstration of the dynamics in the NCO model is shown in Fig. 1. At time $t = 0$, nodes C, D, and E form a stable cluster. No matter what the opinion is of the nodes outside the cluster, the community support inside the cluster is enough for nodes C, D, and E to keep their opinions and not to be invaded by the opposite opinion, which is like a trapped liquid (pore) in TIP. In the stable state, *all* clusters formed by both opinions are stable clusters. Unlike TIP, one opinion in the NCO model can behave at the same time both as an *invading* liquid and as an *invaded* liquid, as expected for opinion dynamics, so the NCO model can demonstrate several unique properties, as we discuss later.

We perform simulations of the NCO model in network models and in a real-life network. The network models include Erdős-Rényi (ER) networks, scale-free (SF) networks [10], and also two-dimensional (2D) regular lattices [including hexagonal (HX), square (SQ), and triangular (TR) lattices]. As an example of real-life networks, we analyze the high energy physics (HEP) citations network [11]. ER networks are characterized by a Poisson degree distribution with average degree $\langle k \rangle$. SF networks are characterized by a power-law degree distribution $P(k) \sim k^{-\lambda}$, with $k_{\min} \le k < k_{\max}$. To ensure network compactness, we choose $k_{\min} = 2$. We use the known natural cutoff for $k_{\max}$ [12]. Simulations on the SF network with $\lambda = 2.5$ [13] are reported here.

Next, we show the emergence of a phase transition in the stable state as the minority opinion σ_- becomes more and more influential (increasing concentration), even well before becoming the majority. We denote by F the fraction of σ_- nodes and S the sizes of the clusters formed by the σ_- nodes in the stable state. The size of the largest and second largest clusters are denoted by S_1 and S_2, respectively, and we define $s_1 \equiv S_1/N$ and $s_2 \equiv S_2/N$. The system reaches a stable state after a few simulation steps [14]. In Fig. 2, we show F, s_1, and s_2 as a function of f for four different networks. We find that F is a monotonically increasing function of f with symmetry around $(f, F) = (0.5, 0.5)$—as expected, since the two opinion states are symmetrical. At a certain critical value $f = f^*$, s_1 shows a sharp increase from a very small value to a *finite* fraction of the entire system, while s_2 displays a sharp peak, a characteristic of a second-order phase transition [15].

The values of f^* and $F(f^*)$ depend only on the type of the network and are almost independent of N. As can be seen in Fig. 2, in the stable state, if the final concentration of σ_- opinion node is above the threshold $F(f^*)$, the σ_- nodes will be able to form a large spanning cluster of the order of the system size N. Below this threshold only isolated small clusters can form. Note that f^* for ER, SF, and HEP networks are all less than 0.5, implying that a phase transition occurs for nodes holding the minority opinion. Only for the HX lattice, $f^* \approx 0.567$ [16].

Note that for the NCO model—in contrast to the social impact model [5]—stable clusters of nodes holding the minority opinion can persistently survive without assuming influential or strong-willed nodes residing inside the clusters.

Next, we present results indicating that the NCO model is in the same universality class as TIP. For regular (site and bond) percolation at criticality, the probability density function of the cluster size S follows a power law $P(S) \sim S^{-\tau}$, where $\tau = 2.055$ for 2D lattices and $\tau = 2.5$ for higher dimensional networks such as ER and SF [17]. In contrast, the TIP model shows a power-law distribution of the sizes of pores [see Fig. 3(a)] with a cumulative distribution function having the form $P(S' > S) \sim S^{1-\tau}$, with $\tau \approx 1.90 \pm 0.01$ for 2D lattices, different from regular percolation.

We find that, for the NCO model in 2D lattices, the cumulative distribution function of cluster sizes at criticality [see Fig. 3(a)] is $P(S' > S) \sim S^{1-\tau}$, with $\tau \approx 1.89 \pm 0.01$, which is close to the τ from the distribution of pore sizes in the TIP model. This fact leads us to hypothesize that for a 2D lattice the NCO model belongs to the same universality class as TIP.

To further test this hypothesis, we study the fractal dimensions of the stable clusters formed by the NCO at criticality. For regular percolation in 2D lattices, the fractal dimensions of the clusters at criticality can be calculated from the power-law relation between S and the cluster diameter, which is represented by either the radius of gyration R_g or the average hopping distances between all pairs of nodes ℓ: $S \sim R_g^{d_f}$ and $S \sim \ell^{d_\ell}$, where $d_f = 91/48 = 1.896$ and $d_\ell \approx 1.678 \pm 0.003$ [17]. For TIP in 2D lattices, the invading liquid has fractal dimensions of $d_f \approx 1.83 \pm 0.01$ and $d_\ell \approx 1.51 \pm 0.01$ [6].

Our simulations for 2D lattices show that the stable clusters of the NCO model at criticality are also fractals. To test whether they have the same fractal dimensions as TIP (as our hypothesis), we plot $R_g/S^{1/d_f}$ and $\ell/S^{1/d_\ell}$ as a function of S in Figs. 3(b) and 3(c). We test different trial values of d_f and d_ℓ to find the best power-law fits for the simulation results. We find that for $d_f = 1.84$, for both SQ and TR, $R_g/S^{1/d_f}$ approaches asymptotically a constant. In contrast, when we choose $d_f = 1.896$ (regular percolation fractal dimension), for $R_g/S^{1/d_f}$ we observe an increasing function with S. On the other hand, when $d_\ell = 1.54$, $\ell/S^{1/d_\ell}$ approaches a constant. In contrast, when $d_\ell = 1.678$ (as regular percolation), $\ell/S^{1/d_\ell}$ is an increasing function of S. We conclude that, for the NCO model, $d_f \approx 1.84 \pm 0.01$ and $d_\ell \approx 1.54 \pm 0.02$, in close agreement with the fractal dimensions of invading fluid in TIP. These results provide further supports for the NCO model belonging to the same universality class as the TIP model.

It is known that for 3D or higher dimensional systems, which include ER and SF, trapping becomes not effective and TIP falls into the same universality class of regular percolation [6]. Our simulations of the NCO model on ER (not shown) and SF networks indeed show the same $P(S' > S)$ [see Fig. 3(a)] and the same fractal dimension (not shown) as for regular percolation. For the NCO model, σ_- nodes serve as both invading liquid and replaced liquid (similarly for the σ_+ nodes), which is the reason why at criticality σ_- nodes have both the d_f and d_ℓ of the invading fluid and the τ of the pores.

To further support the existence of a second-order phase transition, in the insets in Figs. 2(a) and 2(d), we plot $g(S_1, N)$ as a function of N at f^* and $f^* \pm \Delta$, where

$$g(S_1, N) \equiv S_1/N^{\zeta}. \tag{1}$$

Here $\zeta = 0.667$ for high-dimensional networks like SF and ER, and $\zeta = d_f/2$ for 2D lattices [17]. It is another characteristic of a second-order phase transition that at criticality $g(S_1, N)$ approaches a constant. Indeed, for SF with $\lambda = 2.5$ ($f^* \approx 0.450$), when $\zeta = 0.667$, $g(S_1, N)$ approaches a constant. For HX ($f^* \approx 0.567$), when $\zeta = 0.92$, $g(S_1, N)$ also approaches a constant. The fact that the theoretically predicted values of ζ fit well the simulation results provides further evidence for our findings.

To understand why, in contrast to regular percolation [13], the NCO model in the SF network with $\lambda < 3$ and HEP (which is approximately SF) displays a critical percolation behavior at nonzero $F(f^*)$, we study the average degree $\langle k(f) \rangle$ and the cumulative degree distribution $P_f(k' > k)$ of the σ_- nodes in the stable state. Figure 4(a) shows $\langle k(f) \rangle$ for SF networks with $\lambda = 2.5$ and

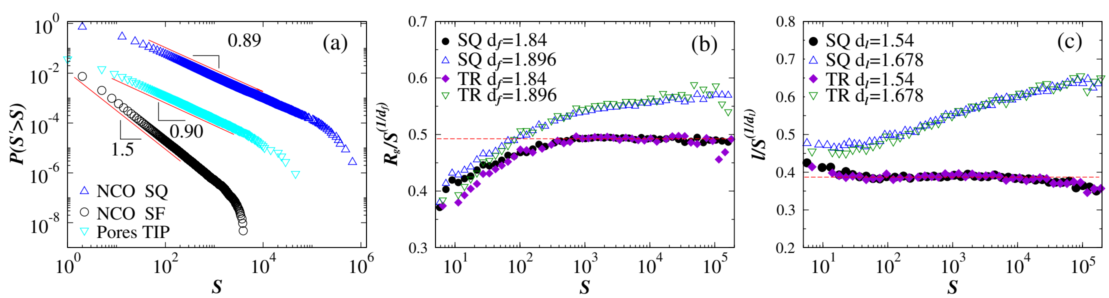

FIG. 3 (color online). (a) The cumulative distribution function of cluster sizes $P(S' > S)$ at f^* for a NCO model on a SQ lattice with $N = 9 \times 10^6$ (3000×3000) and a SF network with $\lambda = 2.5$ and $N = 10^5$. $P(S' > S) \sim S^{1-\tau}$ for both SQ and SF, where $\tau \approx 1.89$ (SQ) and $\tau \approx 2.5$ (SF). $P(S' > S)$ of the sizes of pores of TIP on SQ with $N = 9 \times 10^6$ is also shown, which also takes the form $P(S' > S) \sim S^{1-\tau}$ with $\tau \approx 1.90$. Averages over 100 realizations are shown for all curves. (b) For the NCO model, at criticality, $R_g/S^{1/d_f}$ as functions of S for SQ and TR with $N = 9 \times 10^6$. For the trial value $d_f = 1.84$, $R_g/S^{1/d_f}$ approaches a constant. For $d_f = 1.896$ (regular percolation fractal dimension), $R_g/S^{1/d_f}$ is an increasing function of S. (c) For the NCO model, at criticality, $\ell/S^{1/d_\ell}$ as a function of S for SQ and TR with $N = 9 \times 10^6$. For the trial value $d_\ell = 1.54$, $\ell/S^{1/d_\ell}$ approaches a constant. For $d_\ell = 1.678$ (regular percolation fractal dimension), $\ell/S^{1/d_\ell}$ is an increasing function of S. In (b) and (c), each curve is averaged over 1000 realizations.

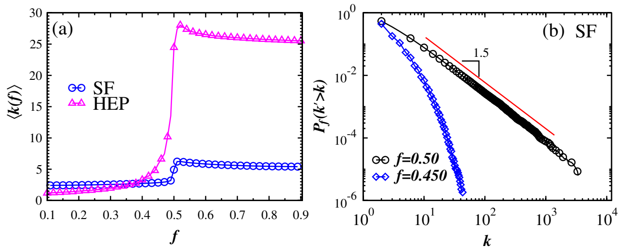

FIG. 4 (color online). (a) The average degree $\langle k(f)\rangle$ of σ_- nodes in the stable state as a function of f for a SF network with $\lambda = 2.5$ and $N = 10^5$ and the HEP network. It is seen that, for $f < 0.5$, $\langle k(f)\rangle$ is significantly smaller than that for $f > 0.5$, since the high degree nodes join the majority opinion [18]. (b) The cumulative degree distribution $P_f(k' > k)$ of σ_- nodes in the stable state for a SF network with $\lambda = 2.5$ and $N = 10^5$. $P_f(k' > k)$ for $f = 0.450$ and 0.5 are plotted. Notice that $f^* \approx 0.450$. The absence of high degree σ_- nodes in a stable state for $f < 0.5$ is again confirmed.

for the HEP network ($\lambda \approx 2.9$). Note that $\langle k(f)\rangle$ shows a significant increase at $f = 0.5$, which demonstrates that the minority opinion nodes have a significantly lower degree compared to those of the majority opinion. We also show $P_f(k' > k)$ for different values of f in Fig. 4(b), which provides further evidence for lower degrees and the appearance of significantly fewer high degree nodes in the minority opinion [18]. This explains why, for SF networks with $\lambda = 2.5$ and for HEP, we observe a phase transition in the NCO model. The minority opinion nodes do not include high degree nodes which are responsible for the formation of large spanning cluster for $p_c \to 0$ in regular percolation. This process is analogous to removing the hubs from a SF network, for which p_c becomes finite [19].

In summary, we propose a nonconsensus opinion model, which allows the stable coexistence of minority and majority opinions. In the stable state, nodes holding the same opinion demonstrate a phase transition from small clusters to large spanning clusters when the concentration of that opinion increases. Our simulations suggest that the phase transition belongs to the same universality class as invasion percolation, which is physically reasonable because, due to the clustering (“community support”) of agents holding the same opinion, stable clusters cannot be invaded by the other opinion (similar to incompressible fluids). Thus, an opinion model can be mapped to a known physics percolation problem.

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[13] We use the SF network with $\lambda = 2.5$. There is no random percolation threshold ($p_c = 0$) for the SF networks with $\lambda < 3$ [12]. In contrast, for the NCO model, we obtain a clear phase transition.
[14] The number of agents who change their opinions decreases exponentially with time. The average number of changes for each node is about 1.5 with standard deviation 0.5 for the SF network with $\lambda = 2.5$.
[15] The critical behavior of the NCO model disappears when applied to networks with a high average degree, because when the average node degree is large, an individual opinion becomes irrelevant and the majority-voter model [3] becomes valid. Thus, the minority opinion will not be able to survive, and the system will eventually reach consensus, unless one introduces a large weight on the node's own opinion.
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