Structure of Correlations in Three Dimensional Spin Glasses

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We investigate the low temperature phase of the three dimensional Edward-Anderson model with Bernoulli random couplings. We show that, at a fixed value Q of the overlap, the model fulfills the clustering property: The connected correlation functions between two local overlaps have power law decay. Our findings are in agreement with the replica symmetry breaking theory and show that the overlap is a good order parameter.

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Spin glasses have unusual statistical properties. In mean field theory, there are intensive quantities that fluctuate also in the thermodynamic limit. This is the effect of the coexistence of many equilibrium states. The correlation functions inside a given state should have a power law behavior: Below the critical temperature, spin glasses are always in a critical state (many glassy systems should share this behavior). These predictions of mean field theory have never been studied in detail, apart from Ref. [1]; the aim of this Letter is to address this point in a careful way.

In order to better characterize the behavior of spin glasses, it is convenient to consider two clones of the same system: $\sigma(i)$ and $\tau(i)$, i being the point of the lattice. The two clones share the same Hamiltonian H_J; the label J indicates the set of random coupling constants.

Let us define the local overlap $q(i) \equiv \sigma(i)\tau(i)$ and the global overlap $q \equiv V^{-1}\sum_i q(i)$, V being the total volume. For the three dimensional Edwards-Anderson (EA) model [2] at zero magnetic field, all simulations confirm that the probability distribution of $P_J(q)$ is nontrivial in the thermodynamic limit; it changes from system to system, its average over the disorder, that we denote as $P(q) \equiv E[P_J(q)]$, is nontrivial, and it has a support in the region from $-q_{\rm EA}$ to $q_{\rm EA}$, $q_{\rm EA}$ being the overlap of two generic configurations belonging to the same state [3]. It is usually assumed that the function $P(q)$ has in the infinite volume limit a delta function singularity at $q = q_{\rm EA}$ that appears as a peak in finite volume systems. In the presence of multiple states, the most straightforward approach consists in identifying the clustering states (i.e., those where the connected correlation functions go to zero at large distance) and to introduce an order parameter that identifies the different states. This task is extremely difficult in a random system where the structure of the states depends on the instance of the system. However, the replica theory is able to make predictions without finding out explicitly the set of states. At this end, the introduction of the two clones is crucial: If the global overlap q has a preassigned value, the correlations of local overlaps $q(i)$ must go to zero at large distances. In other words, q is a good order parameter.

For each realization of the system, we consider two clones. The observables are the local overlaps $q(i)$ and their correlations. We define $\langle O\rangle_Q^J$ as the expectation value of the observable O in the J-dependent Gibbs ensemble restricted to those configurations of the two clones that have global overlap $q = Q$. We define the average expectation values $\langle O\rangle_Q$ as the weighted average over the systems of restricted expectation values:

$$\langle O\rangle_Q = \frac{E[P_J(Q)\langle O\rangle_Q^J]}{E[P_J(Q)]}. \qquad (1)$$

In this Letter, we bring evidence for a main prediction of the replica symmetry breaking theory [4]: The Q-dependent connected correlation functions go to zero when computed in the ensemble $\langle\cdot\rangle_Q$; i.e., the states $\langle\cdot\rangle_Q$ are clustering. The procedure is very similar to the one used in ferromagnetic Ising models (i.e., we consider averages with positive, or negative, total magnetization). The overlap constraint state would be not clustering if the equilibrium state were locally unique (apart from a global change of signs) [5]. Spin glasses are the only known example of a system where the clustering states are labeled by a continuously changing order parameter in the absence of a continuous symmetry (e.g., rotations).

This clustering property has far-reaching consequences: For example, the probability distribution $P(q)_Q^W$ of the window overlaps [6], i.e., the average overlap over a region of size W, becomes a delta function $\delta(Q-q)$ in the infinite volume limit.

We recall some results for the connected correlation functions in the case of short range Ising spin glasses:

$$G(x|Q) = \langle q(x)q(0)\rangle_Q, \quad C(x|Q) = G(x|Q) - Q^2, \qquad (2)$$

and their Fourier transforms $\tilde{C}(k|Q)$. They are obtained by starting from mean field theory and computing the first nontrivial term [7]: They are supposed to be exact in dimensions D greater than 6. Neglecting logarithms, we have in the small k region

$$\begin{aligned} &\tilde{C}(k|Q) \propto k^{-4} \quad \text{for } Q = 0, \\ &\tilde{C}(k|Q) \propto k^{-3} \quad \text{for } 0 < Q < q_{\text{EA}}, \\ &\tilde{C}(k|Q) \propto k^{-2} \quad \text{for } Q = q_{\text{EA}}, \\ &\tilde{C}(k|Q) \propto (k^2 + \xi(Q)^{-2})^{-1} \quad \text{for } Q > q_{\text{EA}}. \end{aligned} \tag{3}$$

The reader may be surprised to find a result for $Q > q_{\text{EA}}$ because the function $P(q)$ is zero in this region in the infinite volume limit. However, for finite systems, $P(q)$ is different from zero for any q, albeit it is very small [8,9] in the region $q > q_{\text{EA}}$. For $Q > q_{\text{EA}}$, an analytic computation of the function $\tilde{C}(k|Q)$ has not yet been done; however, it is reasonable that the leading singularity near to $k = 0$ in the complex plane is a single pole, leading to an exponentially decaying correlation function.

When the dimensions become smaller than 6, we can rely on the perturbative expansion in $\epsilon = 6 - D$ [10]. The predictions at $Q = q_{\text{EA}}$ should not change, and the form of the $k = 0$ singularity at $Q = q_{\text{EA}}$ (i.e., when the two clones belong to the same state) remains k^{-2} as for Goldstone bosons. On the contrary, the $k = 0$ singularities at $Q < q_{\text{EA}}$ should change and

$$\tilde{C}(k|Q) \propto k^{-\tilde{\alpha}(Q)} \quad \text{for } 0 \le Q < q_{\text{EA}}. \tag{4}$$

These perturbative results are the only information we have on the form of $\tilde{\alpha}(Q)$. In the simplest scenario, $\tilde{\alpha}(Q)$ is discontinuous at $Q = 0$ and constant in the region $0 < Q < q_{\text{EA}}$. There is no strong theoretical evidence for the constancy of $\tilde{\alpha}(Q)$ in the region $0 < Q < q_{\text{EA}}$, apart from generic universality arguments. The discontinuity at $Q = 0$ could persist in dimensions not too smaller than 6 and disappear at lower dimensions, as supported by our data in $D = 3$. In the three dimensional case in configuration space, we should have

$$C(x|Q) \propto x^{-\alpha(Q)} \quad \text{for } 0 \le Q \le q_{\text{EA}}, \tag{5}$$

with $\alpha(q_{\text{EA}}) = 1$. For $Q > q_{\text{EA}}$ the correlation should go to zero faster than a power: We tentatively assume that

$$C(x|Q) \propto x^{-1} \exp[-x/\xi(Q)] \quad \text{for } q_{\text{EA}} < Q. \tag{6}$$

In this Letter, we will numerically study the properties of the two overlap connected correlation functions in the three dimensional EA model. The Hamiltonian of the EA model [2] is given by $H_\sigma = -\sum_{|i-j|=1} J_{i,j}\sigma_i\sigma_j$, with $J_{i,j} = \pm 1$ (symmetrically distributed) and Ising spins $\sigma_i = \pm 1$. We have studied cubic lattice systems with periodic boundary conditions of side L for $L = 4, 6, 8, 10, 12, 16$, and 20. The simulation parameters are the same as used in Ref. [11]. We present the results only at temperature $T = 0.7$, while the critical temperature is about $T = 1.11$.

We have first classified the configurations created during the numerical simulations according to the value of the global overlap q. Since the properties of the configurations are invariant under the symmetry $(q \to -q)$, we have classified the configurations into 20 equidistant bins in q^2: For example, the first bin contains all of the configurations where $0 < q^2 < 1/20$. In this way, we compute the correlations $C(x|Q)$. We have measured the correlations only along the axes of the lattice: x is an integer restricted to the range $0, L/2$. As a control we have done the same operation with 10 bins obtaining similar results.

We have first verified that the connected correlations vanish for large systems. At this end, in Fig. 1, we have plotted for $L = 20$ (our largest system) the correlation $G(10|Q)$ versus the average of Q^2 in the bin. The two quantities coincide. The data show strong evidence for the vanishing of the connected two-point correlation function. The prediction of the replica theory is $G(L/2|Q) = Q^2$, neglecting corrections going to zero with the volume.

Further information can be extracted from the data. The analysis of the data should be done in a different way in the two regions $0 \le Q^2 \le q_{\text{EA}}^2$ and $q_{\text{EA}}^2 < Q^2$ as far as two different behavior are expected. In our case q_{EA}^2 can be estimated to be around 0.4.

In the region $0 \le Q^2 \le q_{\text{EA}}^2$, the power law decrease (5) of the correlation is expected. To test this hypothesis [12], we define for each L the quantities $\chi_L^{(s)}(Q)$:

$$\chi_L^{(s)}(Q) = \sum_{x=1}^{L/2} x^s C_L(x|Q), \tag{7}$$

where $C_L(x|Q)$ is the connected correlation function in a system of size L, i.e., $G_L(x|Q) - Q^2$ [in order to decrease the statistical errors we have used the asymptotically equivalent definition $G_L(x|Q) = G_L(x|Q) - G_L(L/2|Q)$]. For large L, $\chi_L^{(s)}(Q)$ should behave as $L^{s+1-\alpha(Q)}$. We have evaluated the previous quantity [more precisely, we have used at the place of $C_L(x|Q)$ its proxy $C_L(x|Q) - C_L(L/2|Q)$ that has smaller statistical errors] for

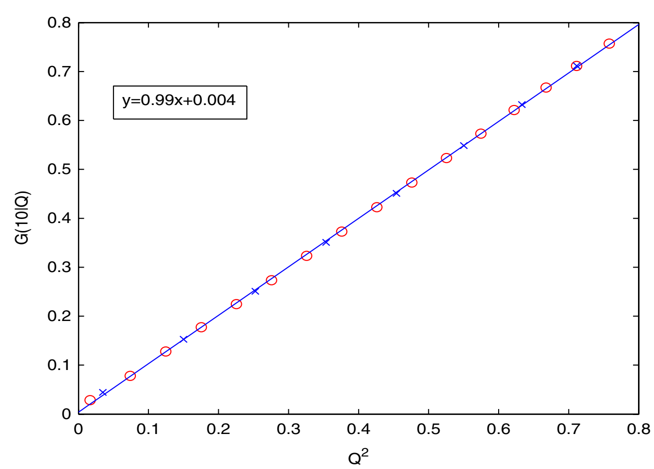

FIG. 1 (color online). The correlation function at distance 10 $G(10|Q)$ averaged in 20 bins (round points) and in 10 bins (crosses) of Q^2 versus the average of Q^2 inside the bin. The data are for a system of size 20, and the straight line is the best fit to the data.

$s = 1, 2$. In the region $Q^2 < 0.4$ we have found that the ratio $\chi_L^{(2)}(Q)/\chi_L^{(1)}(Q)$ is well linear in L. Here the data for $\chi_L^{(s)}(Q)$ can be well fitted as a power of L, and the exponents $\alpha(Q)$ computed using $s = 1$ and $s = 2$ coincide within their errors. These results are no more true in the region $0.5 < Q^2$ indicating that a power law decrease of the correlation is not valid there. The exponents we find with this method are shown in Fig. 2.

In order to check these results for $\alpha(Q)$, we have used a different approach. In the large volume limit, the correlation function should satisfy the scaling $L^{\alpha(Q)}C_L(x|Q) = f(x/L)$. The value of $\alpha(Q)$ can be found by imposing this scaling. At this end for each value of Q we have plotted $L^{\alpha(Q)}C_L(x|Q)$ and found the value of $\alpha(Q)$ for which we get the best collapse. The result of the collapse is shown in Fig. 3 for Q around zero, where for graphical purposes we have plotted $L^{\alpha(Q)}C_L(x|Q)g(x/L)$ versus $\sin(\pi x/L)$, where the function g has been added to compress the vertical scale [we find it convenient to use $g(z) = [1/z + 1/(1-z)]^{-\alpha(Q)}$, following Ref. [13]]. In the left panel we show the collapse using all points with $x \geq 1$, and in the right panel we exclude the correlations at distance $x = 1$. The corresponding values of the exponent are shown in Fig. 2, and they agree with the ones coming from the previous analysis in the region of $Q^2 \leq 0.4$.

The exponent $\alpha(Q)$ is a smooth function of Q^2 which goes to 1 near $Q^2 = 0.4$ in very good agreement with the theoretical expectations. We do not see any sign of a discontinuity at $Q = 0$, and this is confirmed by an analysis with a high number of bins (e.g., 100). However, it is clear that for a lattice of this size value we cannot expect to have a very high resolution on Q, and we should look to much larger lattices in order to see if there is a sign of a building up of a discontinuity and of a plateau. The value of the exponent that we find at $Q = 0$ is consistent with the value

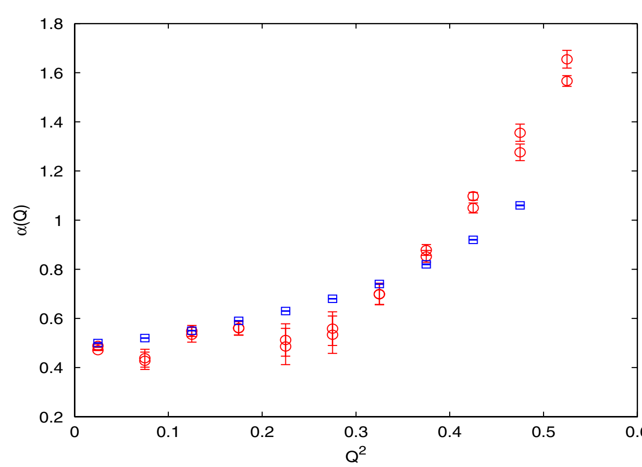

FIG. 2 (color online). Circles are the value of the exponent $\alpha(Q)$ by fitting $\chi^{(s)}(Q)_L$ as a power of L: For each value of Q^2, we show two points corresponding to $s = 1$ and to $s = 2$. Squares represent the same quantity $\alpha(Q)$ as obtained through the scaling approach visible in Fig. 3.

0.4 found from the dynamics [12] and with the value 0.4 found with the analysis of the ground states with different boundary conditions [13].

From the previous analysis it is not clear if the exponent $\alpha(Q)$ has a weak dependence on Q or if the weak dependence on Q is just a preasymptotic effect. In order to clarify the situation, it is better to look to the connected correlations themselves. In Fig. 4, we display the connected correlation $C_L(x|Q)$ as a function of Q^2 for $x = 6, 7, 8, 9$ at $L = 20$, our largest lattice (for the result at $x = 1$, see [14]). We can fit the correlations at fixed L (e.g., $L = 20$) for large x as

$$C_L(x|Q) = A(x, L)[Q^2 - B(x, L)^2], \quad Q^2 < B(x, L)^2, \quad (8)$$

while $C_L(x|Q)$ is very near to zero for $Q^2 > B(x, L)^2$. The goodness of these fits improves with the distance (similar results are valid at smaller L). The value of $B(x, L)^2$ is near to q_{EA}^2, and it is slightly decreasing with L. The validity of the fits (8) for large L would imply that in the region $|q| < q_{\mathrm{EA}}$ the large distance decrease of the correlation function should be of the form $A(x)(Q^2 - q_{\mathrm{EA}}^2)$, and therefore, by comparison with formula (5), the exponent $\alpha(Q)$ should not depend on Q.

However, near $q = q_{\mathrm{EA}}$ we should have a real crossover region. In Fig. 5, we show $C_L(x|Q)g(x)$ at $L = 20$ for $0.475 \leq Q^2 \leq 0.625$ versus $y \equiv [1/x + 1/(2L - x)]^{-1}$ [we use the variable $y = x[1 - O(x/L)]$ to take care of finite volume effects] with $g(x) = (1 - 2x/L)^{-2}$. It seems that the data at $Q^2 > 0.475$ decrease faster than a power at large distances and that the data at $Q^2 = 0.475$ are compatible with a power with exponent -1. It is difficult to extract precise quantitative conclusions, without a careful analysis of the L dependence. We hope that this will be

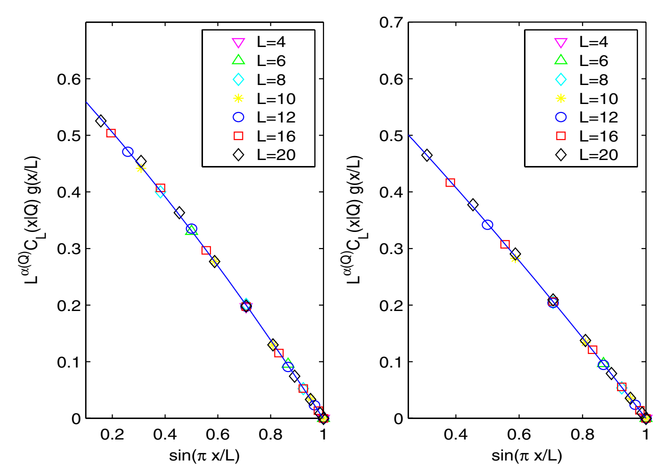

FIG. 3 (color online). The quantity $L^{\alpha(Q)}C_L(x|Q)g(x/L)$ with $g(z) = [1/z + 1/(1-z)]^{-\alpha(Q)}$ versus $\sin(\pi x/L)$ for the set of data in the first bin $Q^2 < 0.5$, using the best value of $\alpha(Q)$. The left panel displays the data for all of the correlations at distances $x \geq 1$ [the corresponding value of $\alpha(Q)$ being 0.50]; in the right panel we have only the data with $x \geq 2$ [the corresponding value of $\alpha(Q)$ being 0.54].

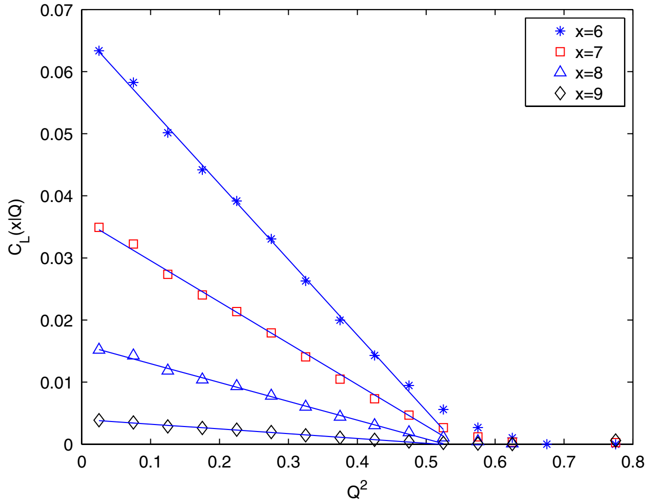

FIG. 4 (color online). The connected correlation $C_L(x|Q)$ as a function of Q^2 for $x = 6, 7, 8, 9$ at $L = 20$. The straight lines are linear fits.

done when the data on the correlation functions on larger lattices are available.

In the region $q_{\mathrm{EA}}^2 \le Q^2$ our task is different: The correlations are short range, and we would like to compute if possible the correlation length. At this end we have fitted the data as

$$C_L(x|Q) = \frac{a}{x+1}\exp[-x/\xi_L(Q)] + (x \longrightarrow L - x) + \mathrm{const}. \quad (9)$$

The choice of the fit is somewhat arbitrary; however, we use it only to check that the correlation length diverges at q_{EA} and that near q_{EA} is well fitted by a $1/x$ power. The fits are good, but this may not imply the correctness of the functional form in Eq. (9). We find that far from $Q = 0.5$

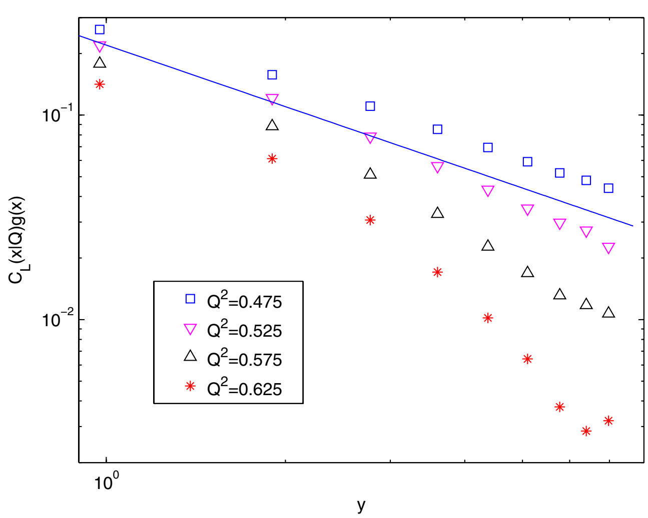

FIG. 5 (color online). Correlation functions at $L = 20$ for $q^2 = 0.475, 0.525, 0.575$, and 0.625 versus $y \equiv [1/x + 1/(2L - x)]^{-1}$. The straight line is proportional to y^{-1}.

the correlation length is independent of L (it is quite small). We have tried to collapse the data for $L > 8$ in the form $\xi_L(Q) = Lf[(Q^2 - q_{\mathrm{EA}}^2)L^{1/\nu}]$. A reasonable collapse has been obtained; however, the q_{EA}^2 is quite small (i.e., 0.25): It is quite possible that there are finite volume effects, and thus different ways to evaluate q_{EA} should give different results on a finite lattice that would converge to the same value in the infinite volume limit.

In conclusion, the global overlap for a two-clone system is a well defined order parameter such that in the appropriate restricted ensemble the two-point connected correlation function decays at large distance. The connected correlations decay as a power whose exponent seems to be independent from Q for $0 \le |Q| < q_{\mathrm{EA}}$: The value of the exponent is in agreement with the results obtained in a different context at $Q = 0$. Moreover, the connected two-point correlation function at $Q = q_{\mathrm{EA}}$ decays like $1/x$ in agreement with the detailed predictions coming from replica theory.

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[3] $q_{\mathrm{EA}} = \max_x q(x)$, where $q(x)$ is the inverse of the cumulative distribution of $P(q)$. Dynamically, $q_{\mathrm{EA}} = \lim_{t\to\infty} \lim_{V\to\infty} E[\langle\sigma_i(t_0)\sigma_i(t_0 + t)\rangle^J]$, where $\sigma_i(t)$ is the spin at point i and at time t.
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