Topological Entanglement and Clustering of Jain Hierarchy States

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We obtain several clustering properties of the Jain states at filling $\frac{k}{2k+1}$: they are a product of a Vandermonde determinant and a bosonic polynomial at filling $\frac{k}{k+1}$ which vanishes when $k+1$ particles cluster together. We show that all Jain states satisfy a ''squeezing rule'' which severely reduces the dimension of the Hilbert space necessary to generate them. We compute the topological entanglement spectrum of the Jain $\nu = \frac{2}{5}$ state and compare it to both the Coulomb ground state and the nonunitary Gaffnian state. All three states have a very similar ''low-energy'' structure. However, the Jain state entanglement ''edge'' state counting matches both the Coulomb counting as well as two decoupled $U(1)$ free bosons, whereas the Gaffnian edge counting misses some of the edge states of the Coulomb spectrum.

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The experimentally observed fractional quantum Hall (FQH) states in the lowest Landau level (LLL) are thought to be described by Laughlin [1] and hierarchy states modeled by Jain's composite fermion wave functions [2]. Jain's states have dramatically large overlap with the true Coulomb ground states, but the process of flux attachment and projection to the LLL renders them hard to analyze (Monte Carlo methods have been devised [3] for treating variants of the Jain states where the projection to the LLL is modified, or simply omitted).

The decomposition of Jain's model states into Slater determinants has not been obtained for $N > 10$ particles [4], and (unlike the Laughlin states) they have not been characterized as unique ground states of some model Hamiltonian. Moreover, their observed large overlap with the ground states of LLL systems with realistic Coulomb interactions is only empirically understood; this has become most evident recently, when other states, with identical filling (and ''shift'') [5,6], as the Jain states, but exhibiting different topological order, have been found to have competitive overlaps with the true Coulomb ground states [7]. Although some of these new states are conjectured to represent gapless critical points [8,9], their large overlap with the Jain states (thought to be gapped in their interior) underscores the need to better understand FQH states from a theoretical standpoint.

In this Letter we describe a previously unrecognized ''clustering property'' of the Jain states which allows them to be (partially) characterized as zero modes of certain pseudopotential Hamiltonians. However (unlike the Laughlin states), they are not unique maximum-density zero modes; while the zero-mode property is insufficient to completely determine the structure of Jain's wave functions, it provides a powerful constraint that enables their numerical construction at significantly larger N. The key technical advance reported here is the identification of the structure of Jain states as ''squeezed polynomials'': they contain only many-body free-particle configurations obtainable from a ''root'' configuration by a two-body operation called ''squeezing,'' defined below. This drastically reduces their Hilbert space dimension.

Armed with this technique, we then investigate the topological entanglement spectrum [10] of the first state in the hierarchy, the $\nu = 2/5$ Jain state for up to $N = 16$ particles, and compare it with both the Coulomb ground state and the so-called Gaffnian state, related to a nonunitary conformal field theory (CFT) [9], which has a Jack polynomial description [6]. We find a virtually identical ''low-energy'' structure in the Schmidt spectral decomposition of these three states. Although the Gaffnian state is very close in both overlap and spectral decomposition to Coulomb and Jain, we directly identify a major difference between these states—the ''edge'' mode structure of the Coulomb entanglement spectrum—and show that it matches the Jain state edge structure as well as that of two $U(1)$ free bosons. We can, however, make no definitive statement on whether the nonunitary Gaffnian state is gapped or gapless [8].

Any fermionic state in the LLL can be written as a product of a Vandermonde determinant and a symmetric polynomial; we first focus on the bosonic variants of model FQH states which omit the Vandermonde factor. We represent an angular momentum partition λ with length $l_\lambda \leq N$ as a (bosonic) occupation-number configuration $n(\lambda) = \{n_m(\lambda), m = 0, 1, 2, \ldots\}$ of each of the LLL orbitals $\phi_m(z) = (2\pi m! 2^m)^{-1/2} z^m \exp(-|z|^2/4)$ with angular momentum $L_z = m\hbar$. $n_m(\lambda)$ is the multiplicity of m in λ. It is useful to identify the ''dominance rule'' [11] (a partial ordering of partitions $\lambda > \mu$) with the ''squeezing rule'' [12] that connects configurations $n(\lambda) \rightarrow n(\mu)$: squeezing is a two-particle operation that moves a particle from orbital m_1 to m_1' and another from m_2 to m_2', where $m_1 < m_1' \leq m_2' < m_2$, and $m_1 + m_2 = m_1' + m_2'$; $\lambda > \mu$ if $n(\mu)$ can be derived from $n(\lambda)$ by a sequence of squeezings. An interacting LLL polynomial P_λ indexed by a root partition λ is defined as a squeezed polynomial if it can be expanded

in occupation-number noninteracting states (monomials m_μ) of orbital occupations $n(\mu)$ obtained by squeezing on the root occupation $n(\lambda)$: $P_\lambda = m_\lambda + \sum_{\mu<\lambda} v_{\lambda\mu} m_\mu$. The $v_{\lambda\mu}$ are rational number coefficients. Partitions λ can be classified by λ_1, their largest part. When any P_λ is expanded in monomials m_μ, no orbital with $m > \lambda_1$ is occupied. P_λ can be interpreted as states on a sphere surrounding a monopole with charge $N_\Phi = \lambda_1$ [13]. A large number of FQH states (such as the Z_k Read-Rezayi sequence [14]) are squeezed polynomials [6].

The root configuration of a squeezed polynomial $n(\lambda)$ has the largest variance: $\Delta\lambda = \sum_{i,j=1}^{N} (\lambda_i - \lambda_j)^2$ of all the partitions $\mu \leq \lambda$. A generic state, such as the ground state of the Coulomb Hamiltonian in the LLL at some arbitrary filling, has nonzero weight on all many-body noninteracting states squeezed from the maximum possible variance $n(\lambda_{\text{Generic State}}) = [\frac{N}{2} 00 \ldots 00 \frac{N}{2}]$, and hence in this case the squeezing property is neither meaningful nor useful. However, for most "model" FQH states, the existence of a root configuration drastically reduces the Hilbert space necessary for generating the state and implies many other special properties of the state.

We now find the root configuration for all bosonic Jain states at filling $k/(k+1)$, defined as the usual composite fermion states at filling $k/(2k+1)$ divided by a Vandermonde determinant. We start with the simplest of these states, the $\nu = 2/3$ state, defined by placing $N/2$ quasiparticles in the Laughlin $1/2$ state and implemented by Jain's operator for t number of quasiparticles [2]:

$$\psi^J_{t\text{qp}} = \text{Det}\begin{pmatrix} \partial_1 & \cdots & \partial_N \\ \vdots & \cdots & \vdots \\ z_1^{t-1}\partial_1 & \cdots & z_N^{t-1}\partial_N \\ 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ z_1^{N-t-1} & \cdots & z_N^{N-t-1} \end{pmatrix} \prod_{i<j}^{N} (z_i - z_j). \quad (1)$$

The single flux attachment Vandermonde factor $\prod_{i<j}^{N}(z_i - z_j)$ is a single Slater determinant of fermionic root configuration $n(\lambda_0) = [111 \ldots 111]$ or $\lambda_0 = (N-1, N-2, N-3, N-4, \ldots, 6, 5, 4, 3, 2, 1, 0)$; one immediately recognizes in λ_0 the powers (angular momentum) of the z_i in the Slater determinant. The determinant operator in Eq. (1), however, has derivative terms, which we denote by $\partial/\partial z = -1$; its root partition in angular momentum basis is $\lambda_{\text{Det}} = (N-t-1, N-t-1, \ldots, 4, 4, 3, 3, 2, 2, 1, 1, 0, 0, -1)$. There are two states at each angular momentum in λ_{Det} because both z^m and $z^{m+1}\partial/\partial z$ operators contained in the determinant have the same angular momentum m. Since the determinant operator now acts on the Vandermonde determinant λ_0, we could immediately add the two angular momentum partitions, but doing this blindly would cause a problem: the resulting partition λ, as it describes a polynomial wave function $\psi_{t\text{qp}}$, must have all its components positive (the final polynomial must be analytic in z's). As such, the last component of λ_{Det} cannot add to the last component of λ_0; adding these two together would correspond to taking the partial derivative $-1 \rightarrow \partial/\partial z$ of a constant $0 \rightarrow z^0$, and the result would vanish. As such, the next maximum variance angular momentum partition one can build is $\lambda = (N-1, N-2, N-3, N-4, \ldots 4, 3, 2, 1, 0) + (N-t-1, N-t-1, \ldots, 4, 4, 3, 3, 2, 2, 1, 1, 0, -1, 0) = (\ldots 14, 13, 11, 10, 8, 7, 5, 4, 2, 0, 0)$, where we have written only the angular momentum close to the north pole in the final partition. When written in occupation number, the root configuration is $n(\lambda) = [201011011011011011 \ldots]$. Creating an $\vec{L} = 0$ state requires that the north pole be identical to the south pole, and hence the root configuration number for the $\nu = 2/3$ state reads $n(\lambda_{\nu=2/3}) = [201011011011 \ldots 0110110110102]$. The bulk occupation configuration contains 2 particles in 3 orbitals (110), as expected for a $\nu = 2/3$ state. For the fermionic state at $\nu = 2/5$, $\psi_{\nu=2/5} = \psi_{\nu=2/3} \prod_{i<j}^{N} (z_i - z_j)$; the root occupation number reads $n(\lambda_{\nu=2/5}) = [11001001010010100101 \ldots 10100101001010010011]$. A similar procedure, starting from the appropriate projection operators in the first k Landau level (LL), allows us to obtain the root occupation number for all Jain states at filling $k/(k+1)$, with the result given in Fig. 1.

The root configuration presented in Fig. 1 for general k allows us to determine part of the Hamiltonian for which the Jain states are exact zero modes. Let k particles cluster at one point, which, by translational invariance, we pick to be the origin. Because all the monomials included in the Jain state are squeezed from $n(\lambda_{\nu=k/(k+1)})$, placing k particles at the origin results in monomials squeezed from $[00101^2 01^3 0 \ldots 1^{k-1} 01^k 01^k \ldots 1^k 01^k 01^{k-1} \ldots 01^3 01^2 010k]$. These monomials are $\sim \prod_{i=k+1}^{N} z_i^2$, and hence the full polynomial vanishes when a $z_{k+1} \rightarrow 0$; since the origin is not special by translational invariance, we have

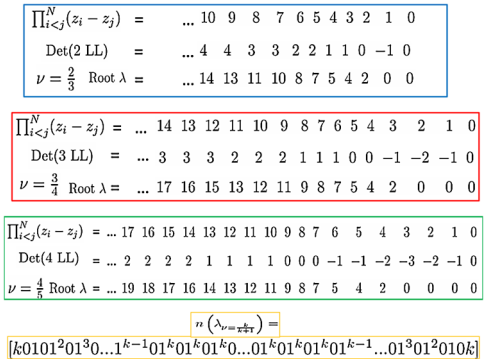

FIG. 1 (color online). Root partition in angular momentum basis for $\nu = \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \ldots \frac{k}{k+1}$ states can be written as the sum of the Vandermonde determinant partition plus the maximum root partition of the determinant operator of k LL projected to the LLL. The root occupation configuration contains k particles in $k+1$ orbitals $[1^k 0]$ when deep in the bulk. Close to the north and south pole there are deviations from this rule.

$$\psi^J_{\nu=k/k+1}(z_1=Z,\ldots,z_k=Z,z_{k+1},\ldots,z_N)\sim\prod_{i=k+1}^{N}(Z-z_i)^2.$$

$\psi^J_{\nu=k/(k+1)}$ are zero modes of the pseudopotential V^0_{k+1}, which eliminates the $\vec{L}=0$ state of a $k+1$-body cluster.

Unfortunately, the above Hamiltonian and root partition do not uniquely define the Jain states. Imposing the $\vec{L}=0$ condition on the squeezed polynomial with the Jain root partition $n(\lambda_{\nu=k/(k+1)})$ results in a conjectured $E[(N+2)/4]$ linearly independent $\vec{L}=0$ polynomials (where $E[x]$ is the integer part of x). We pick the simplest state at bosonic $\nu=2/3$ or fermionic $\nu=2/5$ to analyze further. From now on, we return to the fermionic state. We reduced the problem of determining the order $N!$ Slater determinants in the decomposition of a Jain state to that of determining $E[(N+2)/4]$ constants. While the usual Monte Carlo integration procedures would fail to accurately compute the full decomposition, they may be used to determine the components of the Jain state on this reduced basis. With this method, we are able to obtain the Jain state for up to $N=16$ particles on the sphere geometry (the dimension of the squeezed Hilbert space is 99 608 768, compared to the original full size 155 484 150) with 0.9999 accuracy; the previous largest size was $N=10$ particles [4].

We also constructed the nonunitary Gaffnian state [9] for $N=16$ particles (squeezed Hilbert space dimension = 91 736 995), uniquely defined as the $\vec{L}=0$ squeezed polynomial with bosonic root occupation $n(\lambda_{\text{Jack }\nu=2/3})=[2002002002\ldots2002002]$ [6] multiplied by a Vandermonde determinant. In Fig. 3 we show the overlap of both the Gaffnian and the Jain state with the ground state of Coulomb plus delta function δV_1 interaction obtained by exact diagonalization (the large overlap of the Coulomb state with the Gaffnian for up to $N=12$ particles was previously noted in [9]). The overlap is above 95% for both states for $\delta V_1 > -0.06$. There is a phase transition at $\delta V_1 \approx -0.08$.

In order to better understand the remarkably large overlap, topological order, and the differences between the Jain and Gaffnian ground state, we compute their topological entanglement spectrum. On the sphere (they have identical filling and shift), we cut the state into two hemisphere blocks A and B. Following [10], we introduce the entanglement spectrum ξ as $\lambda_i=\exp(-\xi_i)$, where λ_i are the eigenvalues of the reduced density matrix ρ_A of one hemisphere. The eigenvalues can be classified by the number of fermions N_A in the A block, and also by the total angular momentum $L_z^{(A)}$ of the A block. In a topological state, the low-lying spectrum ξ_i of the reduced density matrix for fixed N_A, plotted as a function of $L_z^{(A)}$, should display a structure reflecting the CFT describing the edge physics and should be separated by a gap from a higher "non-CFT" part of the spectrum. In Fig. 2, this CFT spectrum is defined as every ξ below the horizontal line at $\xi\approx 8$ (below the light blue line). This was shown to be the case for the $\nu=5/2$ state [10] as well as for the Laughlin $\nu=1/3$ state [15]. In our case, $\nu=2/5$, the entanglement gap is not extremely apparent. The three states, Gaffnian, Jain, and Coulomb, have the same low-energy entanglement structure as can be seen in Fig. 2. The counting of entanglement eigenvalues for the Gaffnian at a certain angular momentum $L_z^{(A)}$ is seen to correspond to the counting of occupation-number configurations of angular momentum $L_z^{(A)}$ satisfying the generalized Pauli principle [6] of not more than 2 particles in 5 consecutive orbitals and, by virtue of being fermions, not more than 1 particle in each orbital. The counting of edge modes reads 1,1,3,5,10..., the same as that obtained by different methods in [16]. The Jain state has a very similar "low-energy" entanglement structure with the Gaffnian state, but also exhibits extra higher energy levels not present in the Gaffnian. We remark that the Jain state, not being a pure CFT state (i.e., not obtained as a correlator of CFT primary fields, but rather of their derivatives [17]), has an entanglement gap of its own. Some of the spectral levels present in the Jain entanglement spectrum are nongeneric and should become clearly gapped in the thermodynamic limit. (E.g., at $L_z^{(A)}=80$ the entanglement spectrum is formed by one low-lying eigenvalue and other high-energy ones with very little weight in the Jain state. The difference between these values seems to define an entanglement gap for the Jain state itself.) The presence of an entanglement gap in the Jain state differentiates it from "pure" CFT states, and makes the counting of the edge-state spectrum difficult. To

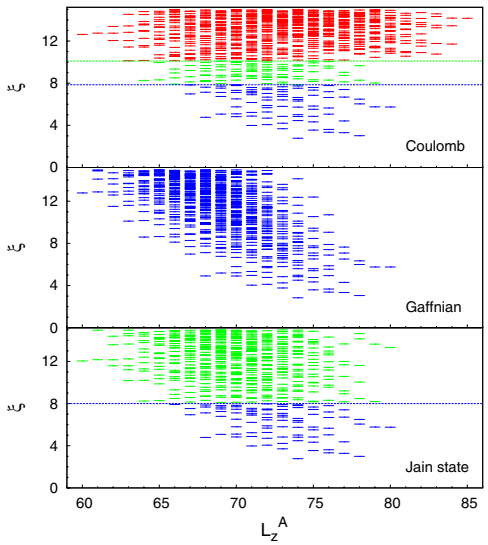

FIG. 2 (color online). Top: Topological entanglement of the pure Coulomb at $\nu=\frac{2}{5}$. The levels below the light blue line at $\xi\approx 8$ are almost identical to the Gaffnian levels, whereas the levels below the green line are almost identical to those of the Jain state, to within 0.003%–3%. Middle: Topological entanglement spectrum of the Gaffnian state. Bottom: Topological entanglement spectrum of the $N=16$, $\nu=2/5$, Jain state. The low-energy structure of the Jain state (in blue at $\xi\approx 8$) is almost identical to that of the Gaffnian (below the blue line).

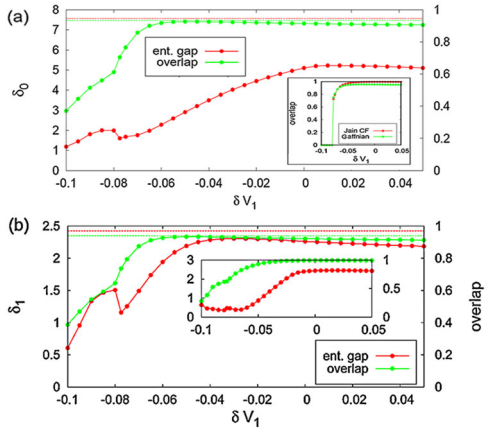

FIG. 3 (color online). *Red plots [starting at $\delta_0 \approx 1$ in (a) and $\delta_1 \approx 0.5$ in (b)]*: Entanglement gap of the Coulomb state for $L_z^{(A)} = 80$ (a), $L_z^{(A)} = 79$ (b) as a function of added hard-core potential. The entanglement gap is discontinuous (a, b) at similar values of δV_1 for which the overlap of the Gaffnian and Jain with the Coulomb ground state collapses [see inset (a): overlaps of both Gaffnian and Jain states with the Coulomb ground state as a function of added hard-core interaction δV_1 for $N = 16$ particles]. A phase transition occurs close to $\delta V_1 \simeq -0.08$.] The red dashed line is the entanglement gap for the Jain state. *Green plots [starting at $\delta_0 \approx 3$ in (a) and $\delta_1 \approx 1$ in (b)]*: Overlap of the reduced density matrix eigenstates for each of the $L_z^{(A)} = 80$, 79 between Coulomb and Gaffnian. The green dashed line is the calculation result between Jain and Gaffnian. Inset (b): Similar results in the two $U(1)$ free boson sector.

proceed, we count only the eigenvalues of the Jain state that match the eigenvalues of the Coulomb spectrum (below the horizontal green line at $\xi \approx 10$ in Fig. 2). This should provide us with the "universal" counting of edge states for a finite size system. As seen in Fig. 2, this counting is 1:2:5 for $\Delta_L = 0$, 1, 2 units away from $L_z^{(A)} = 80$. This matches the counting of two $U(1)$ free bosons as predicted by the hierarchy construction.

The Coulomb state follows most of the low-energy eigenvalues of the Gaffnian (up to $\xi = 8$) and Jain state (up to $\xi = 10$). While two states that have almost identical spectral decomposition necessarily have large overlap, the converse is not true, as large overlap can be accidental. The almost identical low-energy spectral decompositions indicate that the large overlap is not accidental, which is puzzling as the Gaffnian and Jain states represent different states of matter. As in [10], we denote the gap between the lowest two ξ_i, at the $L_z^{(A)}$ value where the highest-$L_z^{(A)}$ member of the CFT spectrum occurs, as δ_0. In Fig. 3, this is the gap between the lowest two states at $L_z^{(A)} = 80$. We define the quantities δ_1 as the gaps at $L_z^{(A)} = 79$ values between the values of the ξ_i's for the CFT state and the next Coulomb value. As noted previously, the Jain state has its own entanglement gap, equal at $L_z^{(A)}$ with the difference between the lowest $\xi_i \approx 6$ and the next one at $\xi_i \approx 13.5$. We study the evolution of the entanglement gaps $\delta_{0,1}$ as we tune the interaction across a phase transition. In Fig. 3, we plot δ_0 as a function of the pseudopotential δV_1 for the $\nu = 2/5$ case, which shows a dramatic decrease of the "entanglement gap" around the region of the phase transition. For values of $\delta V_1 < 0.08$ the CFT-like structure of the entanglement spectrum is lost. Entanglement gaps can also be computed in the two $U(1)$ free boson sectors (see insets of Fig. 3). The gap at $L_z = 79$ between the second Jain (green, below the $\xi \approx 10$ horizontal line) eigenvalue and the next Coulomb (red, above the $\xi \approx 10$ horizontal line) eigenvalue, δ_1, becomes negligible for values of δV_1 larger than those involved in the Gaffnian sector. We notice that for $N = 12$, this is correlated to the first excited state having its angular momentum changing from $L = 6$ to $L = 2$ (around $\delta V_1 = -0.06$).

In conclusion, we showed that the Jain states at $\nu = k/(2k+1)$ are zero modes (but not highest density) of a $k+1$-body pseudopotential and exhibit a Hilbert space size reducing squeezing property. We analyzed the entanglement spectrum of the Jain, Coulomb, and nonunitary Gaffnian states at $\nu = 2/5$ and found a similar low-energy structure which proves their large overlap is not accidental; still, the Jain state contains some physical low-energy levels not present in the Gaffnian.

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[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[3] R. K. Kamilla and J. K. Jain, Int. J. Mod. Phys. B **11**, 2621 (1997).
[4] G. Dev and J. K. Jain, Phys. Rev. B **45**, 1223 (1992).
[5] S. H. Simon *et al.*, Phys. Rev. B **75**, 075317 (2007).
[6] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).
[7] N. Regnault, M. O. Goerbig, and Th. Jolicoeur, Phys. Rev. Lett. **101**, 066803 (2008).
[8] N. Read, Phys. Rev. B **79**, 045308 (2009).
[9] S. H. Simon, E. H. Rezayi, and N. R. Cooper, Phys. Rev. B **75**, 075318 (2007).
[10] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[11] R. P. Stanley, Adv. Math. **77**, 76 (1989).
[12] B. Sutherland, Phys. Rev. A **4**, 2019 (1971).
[13] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[14] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[15] O. Zozulya, M. Haque, and N. Regnault, Phys. Rev. B **79**, 045409 (2009).
[16] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **101**, 246806 (2008); Phys. Rev. B **77**, 184502 (2008); Phys. Rev. Lett. **102**, 066802 (2009).
[17] H. Hansson *et al.*, Phys. Rev. Lett. **98**, 076801 (2007); Phys. Rev. B **76**, 075347 (2007).