Electromagnetic Field Quantization in Time-Dependent Linear Media

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We present a quantization scheme for the electromagnetic field in time-dependent homogeneous nondispersive conducting and nonconducting linear media without sources. Using the Coulomb gauge, we demonstrate this quantization can be mapped into a damped (attenuated) time-dependent quantum harmonic oscillator. Remarkably, we find that the time dependence of the permittivity, for $\dot{\varepsilon} > 0$, gives rise to an attenuation of the radiation field. Afterwards, we obtain the exact wave functions for this problem and consider an exponential time accretion of the permittivity as a particular case.

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In order to investigate the quantum properties of the electromagnetic field, we need to quantize the phenomenological Maxwell Equations. In introductory quantum optics textbooks [1,2], this quantization is traditionally performed in empty cavities or in free space, associating a quantum-mechanical harmonic oscillator with each mode of the radiation field. Recently, the quantization of light propagating through material media has drawn special attention of physicists [3–9], motivated by the growth of experiments on quantum optics processes taking place inside material media [6,8,9]. Several approaches have been employed to tackle the problem of the quantization of the electromagnetic field inside material media [3–9], which have mainly considered materials with electric permittivity [5,6,10–14]: (1) real (nondispersive) and either homogeneous or inhomogeneous media; (2) real and dependent on time and position (time-dependent nonuniform media); or (3) position and frequency dependent (dispersive inhomogeneous media). It is worth remarking that, in dispersive media, the inclusion of losses into the system is, in general, introduced by a reservoir (a continuum of harmonic oscillators) which leads to an energy flow from the medium to the reservoir [8,9]. On the other hand, the problem of conducting media has also been considered [15–19]. Particularly, the case with real and constant conducting properties has been discussed in [15,19], while the quantization in nondispersive conducting media with time-dependent parameters and without current distribution was studied in [18]. In these studies, the losses were introduced phenomenologically as a set of time-dependent parameters.

Although there are various articles discussing the quantization of the electromagnetic field in the presence of time-dependent dielectric properties and a few in the presence of time-dependent conducting media, to our knowledge, none of them considered both properties simultaneously. Moreover, the effects of the time-dependence of the electric permittivity is barely mentioned. As we will see later, this time-dependence gives rise to an attenuation of the radiation field, similarly to the one which appears in dispersive time-independent media. This notable property, to the best of our knowledge, has not been discussed in the literature. This time-dependence is of great interest for applications such as in nuclear [17] and plasma physics (Unruh effect and dynamical Casimir effect) [20–22].

In this Letter, we present a simple and comprehensive unified phenomenological quantization scheme of the electromagnetic field in homogeneous nondispersive linear media, in the absence of sources, with time-dependent electric permittivity and conductivity. Similarly to the case of empty cavities, we demonstrate that this quantization can be performed by associating a damped or attenuated quantum-mechanical time-dependent harmonic oscillator with each mode of the quantized field. We also derive the exact Schrödinger wave functions for this problem and consider a special case of an exponential time accretion of the permittivity.

To begin with, we write the Maxwell's equations for the electromagnetic field in a homogeneous conducting linear media in the absence of charge sources with time-dependent parameters

$$\nabla \cdot \mathbf{D} = 0, \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}, \tag{4}$$

where $\mathbf{D} = \varepsilon(t)\mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$, and $\mathbf{J} = \sigma(t)\mathbf{E}$. Here, $\varepsilon(t)$ and $\sigma(t)$ are heuristically introduced as the time-dependent electric permittivity and conductivity, respectively, while μ_0 is the magnetic permeability. In general, the electric permittivity and the magnetic permeability are complex; however, we will restrict our discussion to materials where they are real. This is the case [23,24], for instance, of poor conductors and other materials for frequencies below the resonant frequency.

In quantum optics, a convenient gauge choice is the Coulomb gauge [1,2], for which the divergence of the vector potential $\mathbf{A}$ is zero and the scalar potential is null in the absence of sources. Consequently, both the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields are determined from the vector potential as

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{and} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}. \tag{5}$$

It is worth mentioning that in the Coulomb gauge, the vector potential is purely transverse [1,2]. Therefore, it is easy to verify that it obeys the damped wave equation

$$\nabla^2 \mathbf{A} - \mu_0(\dot{\varepsilon} + \sigma)\frac{\partial \mathbf{A}}{\partial t} - \mu_0 \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0, \tag{6}$$

where the dot represents a time derivative. From Eq. (6), we notice the appearance of the unusual term $\dot{\varepsilon}(t)$. Consequently, the time dependence of the electric permittivity, which, in principle, may be associated with the internal response of the localized charges to an external perturbation, causes an additional attenuation of the radiation field (for $\dot{\varepsilon} > 0$). What is more, for nondispersive nonconducting dielectric media, $\sigma(t) = 0$, the media become absorbing because of the time dependence, just as if it were in contact with a reservoir (time-dependent background medium) or if it were a conductor. We will come back to this point later. Now, we turn our attention to the solutions of Eq. (6).

Using the well-known separation of variables method, we write the vector potential in terms of the mode $\mathbf{u}_l(\mathbf{r})$ and amplitude $q_l(t)$ functions of each cavity mode [1,2,15,19] as

$$\mathbf{A}(\mathbf{r}, t) = \sum_l \mathbf{u}_l(\mathbf{r}) q_l(t). \tag{7}$$

The substitution of Eq. (7) into the damped wave Eq. (6) leads to

$$\nabla^2 \mathbf{u}_l(\mathbf{r}) + \frac{\omega_l^2}{c_0^2}\mathbf{u}_l(\mathbf{r}) = 0, \tag{8}$$

$$\frac{\partial^2 q_l}{\partial t^2} + \frac{\dot{\varepsilon} + \sigma}{\varepsilon}\frac{\partial q_l}{\partial t} + \Omega_l^2(t) q_l = 0, \tag{9}$$

where ω_l is the natural frequency of the mode l, $c_0 = 1/\sqrt{\mu_0 \varepsilon(0)}$ is the velocity of light inside the medium at $t = 0$ and $\Omega_l(t)$ is a modified frequency defined as

$$\Omega_l = \frac{c(t)\omega_l}{c_0}, \tag{10}$$

with $c(t) = 1/\sqrt{\mu_0 \varepsilon(t)}$ being the velocity of the electromagnetic wave in the time-dependent medium.

The equations of motion for the amplitudes $q_l(t)$ given by Eq. (9) can be directly obtained from the classical Hamiltonian

$$H_l(t) = e^{-\Lambda(t)}\frac{p_l^2}{2\varepsilon_0} + \frac{1}{2}e^{\Lambda(t)}\varepsilon_0 \Omega_l^2(t) q_l^2, \tag{11}$$

where q_l and p_l are canonical conjugated variables, with $\Lambda(t)$ given by

$$\Lambda(t) = \int_0^t \frac{\dot{\varepsilon}(\tau) + \sigma(\tau)}{\varepsilon(\tau)} d\tau. \tag{12}$$

Hence, the total Hamiltonian of the electromagnetic field is a sum of individual Hamiltonians corresponding to each mode, that is, $H = \sum_l H_l$.

In the following discussion, we will use the equation of motion for the mode functions, Eq. (8), and the Hamiltonians, Eq. (11), to obtain a quantum description of the electromagnetic field. Let us first focus on Eq. (8). Considering the electromagnetic field to be contained in a certain cubic volume V of nonrefracting media, the mode functions are required to satisfy the transversality condition, $\nabla \cdot \mathbf{u}_l(\mathbf{r}) = 0$ and to form a complete orthonormal set [1,2]

$$\int_V \mathbf{u}_l^* \mathbf{u}_{l'} d^3\mathbf{r} = \delta_{ll'}, \tag{13}$$

where the integral is performed over the volume V. Further, assuming periodic boundary conditions on the surface, the mode functions may be written in term of plane waves as [1,2,19]

$$\mathbf{u}_{l\nu}(\mathbf{r}) = L^{-3/2} e^{\pm i \mathbf{k}_l \cdot \mathbf{r}} \hat{\mathbf{e}}_{l\nu}, \tag{14}$$

where $L = V^{1/3}$ is the size of the cube, $|\mathbf{k}_l| = \omega_l/c_0$ is the wave vector, and $\hat{\mathbf{e}}_{l\nu}$ are unit vectors in the directions of polarization ($\nu = 1, 2$), which must be perpendicular to the wave vector because of the transversality condition. Furthermore, the periodic boundary conditions allow us to write $\mathbf{k}_l = (2\pi/L)(l_1\hat{\mathbf{e}}_x + l_2\hat{\mathbf{e}}_y + l_3\hat{\mathbf{e}}_z)$ where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{e}}_z$ are unit vectors in the directions of x, y, and z, respectively, and l_1, l_2, and l_3 are integers.

Now that the spatial mode functions are completely determined, we move our attention to the canonical operator $q_l(t)$ in order to obtain the vector potential. For this purpose, we solve the Schrödinger equation associated with the Hamiltonian (11)

$$H_l(t)\Psi[q_l(t), t] = i\hbar \frac{\partial}{\partial t}\Psi[q_l(t), t], \tag{15}$$

where the canonical momentum becomes the momentum operator $p_l = -i\hbar\partial/\partial q_l$, with $[q_l, p_l] = i\hbar$. We obtain the solutions of Eq. (15) with the aid of the dynamical invariant method devised by Lewis and Riesenfeld [25,26]. According to this method, if the system admits an exact invariant $I(t)$ (constant of motion), it is possible to find the complete quantum state, whose evolution is given by the Schrödinger equation, in terms of the eigenstates $\phi_\lambda(q_l, t)$, with time-independent eigenvalues λ, of this invariant operator and a phase factor, $\mu_\lambda(t)$. Actually, the quantum states may be written as

$$\Psi_\lambda(q_l, t) = e^{i\mu_\lambda(t)}\phi_\lambda(q_l, t), \tag{16}$$

with the orthonormality condition $\langle \phi_{\lambda'} | \phi_\lambda \rangle = \delta(\lambda' - \lambda)$

and the phase function $\mu_\lambda(t)$ given by

$$\hbar \frac{d\mu_\lambda(t)}{dt} = \langle \phi_\lambda | i\hbar \frac{\partial}{\partial t} - H_l | \phi_\lambda \rangle. \tag{17}$$

Further, the invariant $I(t)$ must not contain time derivative operators and satisfy the condition

$$\frac{dI_l(t)}{dt} = \frac{1}{i\hbar}[I_l, H_l] + \frac{\partial I_l}{\partial t} = 0. \tag{18}$$

One such invariant is the linear Hermitian operator [19,27]

$$I_l(t) = \alpha_l(t) q_l + \beta_l(t) p_l + \gamma_l(t), \tag{19}$$

where $\alpha_l(t)$, $\beta_l(t)$, and $\gamma_l(t)$ are time-dependent real functions to be determined. From Eqs. (18) and (19), it is easy to verify that $\beta_l(t)$ obeys the classical equation of motion of a time-dependent damped harmonic oscillator

$$\ddot{\beta}_l(t) + \frac{\dot{\varepsilon} + \sigma}{\varepsilon} \dot{\beta}_l(t) + \Omega_l^2(t) \beta_l = 0, \tag{20}$$

while $\gamma_l(t)$ is a constant. Setting, without loss of generality, this constant at zero, the linear invariant can be cast into the form

$$I_l(t) = \beta_l(t) p_l - \varepsilon_0 e^{\Lambda(t)} \dot{\beta}_l(t) q_l. \tag{21}$$

Hence, it is straightforward to show that the normalized eigenstates of $I_l(t)$ are

$$\phi_l(q_l, t) = \sqrt{\frac{1}{2\pi\hbar\beta_l(t)}} \exp\left[\frac{i}{\hbar\beta_l(t)} \times \left(\frac{\varepsilon_0 \dot{\beta}_l(t) e^{\Lambda(t)}}{2} q_l^2 + \lambda q_l\right)\right]. \tag{22}$$

On the other hand, after some basic calculation, we get the phase function

$$\mu_\lambda(t) = -\frac{\lambda^2}{2\hbar\varepsilon_0} \int_0^t \frac{e^{-\Lambda(\tau)}}{\beta_l^2(\tau)} d\tau. \tag{23}$$

Therefore, we can write the solutions of the Schrödinger equation as

$$\Psi_\lambda(q_l, t) = \sqrt{\frac{1}{2\pi\hbar\beta_l(t)}} e^{i\mu_\lambda(t)} \times \exp\left[\frac{i}{\hbar\beta_l(t)} \times \left(\frac{\varepsilon_0 \dot{\beta}_l(t) e^{\Lambda(t)}}{2} q_l^2 + \lambda q_l\right)\right], \tag{24}$$

and a general state is given by a superposition of these states

$$\Psi(q_l, t) = \int_{-\infty}^{\infty} g(\lambda) \Psi_\lambda(q_l, t) d\lambda, \tag{25}$$

where $g(\lambda)$ is the weight function which determines the state of the system.

In the following, we present an example of a specific functional dependence of the permittivity on time. We will restrict our discussion to dielectric materials [$\sigma(t) = 0$], but the results may be readily generalized to conducting materials. Particularly, we will consider an increasing exponential dependence of the form [17]

$$\varepsilon(t) = \varepsilon_0 e^{\eta t}, \tag{26}$$

where η is a positive constant. For this case, Eq. (20) reduces to

$$\ddot{\beta}_l(t) + \eta \dot{\beta}_l(t) + \omega_l^2 e^{-\eta t} \beta_l = 0, \tag{27}$$

whose solution is known to be [28]

$$\beta_l(t) = e^{-\eta t/2}\left[A J_1\left(\frac{2\omega_l}{\eta} e^{-\eta t/2}\right) + B Y_1\left(\frac{2\omega_l}{\eta} e^{-\eta t/2}\right)\right], \tag{28}$$

where J_1 and Y_1 are Bessel functions of first and second kind, respectively, and A and B are constants. Here, it is worth remarking that some proposals have been investigated in which the physical properties of the medium inside the cavity change with time. In particular, Yablonovitch [20] has used a model for plane wave in a medium with refracting index decreasing in time to study the Unruh (observers in an accelerating reference frame see a thermal radiation field at a temperature proportional to their acceleration) [21] and Casimir (attractive interaction between two perfectly conducting plates separated by a short distance in the vacuum) [22] effects. Further, some authors have investigated material media with negative permittivity and permeability in some range of frequencies. This kind of medium is usually called left handed, since the electric field, the magnetic field, and the wave vector of a plane wave form a left-handed system in this range of frequencies [29–31]. Moreover, the observation of negative index in recent years has attracted a lot of attention of physicists to left-handed materials. Also, recently, Alsing *et al.* [32] have studied ion trap simulations in a de Sitter universe in which the phonon excitation of ions in a trap, with a trap frequency exponentially modulated at a constant rate, exhibits the usual Unruh effect for uniformly accelerated detectors. Hence, these subjects are possible applications of the time-dependent model of Eq. (26). Obviously, our formalism may also be applied to time-dependent permittivity with exponential decrease and other forms of time dependence.

We are now ready to find the electric field confined in the cubic volume of side L and analyze some of its quantum properties. First of all, the electric field operator can be written as

$$\mathbf{E}(\mathbf{r}, t) = \frac{e^{-\eta t}}{\varepsilon_0 L^{3/2}} \sum_l \sum_{\nu=1,2} \hat{\mathbf{e}}_{l\nu} e^{\pm i \mathbf{k}_l \cdot \mathbf{r}} p_l(t). \tag{29}$$

Then, from Eqs. (24) and (25), we can build Schrödinger states to investigate quantum properties of the electromagnetic field. In fact, for a specific $g(\lambda)$, we can construct Gaussian wave packet [19] states and coherent squeezed states [33]. Here, let us remember that the Hamiltonian (11), which can be written in terms of the creation and annihilation operators [34], belongs to the class of Hamiltonian which generates coherent or squeezed states [35]. Moreover, one can readily verify that, for a Gaussian wave packet state, the expectation value of the electric and magnetic field operators propagating along any direction vanishes. Thus, these states do not describe the classical behavior of the electromagnetic field. In contrast, if one calculates the expectation value of these operators in a coherent or squeezed state, the results look like the classical fields, that is, the expectation values describe the evolution of a classical field [36]. This result agrees with the original Schrödinger's idea about coherent states. Evidently, we also can use the above-mentioned states to study other quantum properties of the electromagnetic field, such as, the quantum fluctuations, quantum correlations, and uncertainty product for each mode of the quantized field.

In summary, we have presented a direct and simple quantization scheme for the electromagnetic field in conducting and nonconducting media with time-dependent parameters. We have shown that this quantization can be performed by associating a damped quantum-mechanical harmonic oscillator with each mode of the radiation field. As a consequence, we have established an unification of the procedure to quantize electromagnetic waves in empty cavities (or free space) and cavities filled with a (time-dependent) material medium. In the former case, it is usually performed by the association of an ordinary oscillator with each mode of the quantized field, and in the latter one it can be performed by associating a time-dependent harmonic oscillator. Therefore, contrary to the common belief that one cannot associate an harmonic oscillator to each mode of the radiation field in the presence of a lossy background medium [8,9], this association is in fact not only possible but also useful to treat time-dependent nondispersive media. Another important result of our work concerns the electric permittivity time variation. We have shown that this dependence, even in the absence of dispersion, introduces losses into the system similarly to conducting media or time-dependent background medium. This remarkable and intriguing effect has not been explored in the literature yet. Finally, we argue that the approach developed in this contribution can be useful to investigate subjects related to the quantization of light propagating in conducting or nonconducting media with material properties varying in time as, for example, the well-known Fabry-Perot cavity [37].

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[1] H. W. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York, 1993).
[2] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1995).
[3] R. J. Glauber, *Quantum Theory of Particle Trapping by Oscillating Fields* (Lyman Laboratory of Physics, Harvard University, Cambridge, MA 02138, USA, 1991).
[4] B. Huttner and S. M. Barnett, Phys. Rev. A **46**, 4306 (1992).
[5] R. Matloob, R. Loudon, S. M. Barnett, and J. Jeffers, Phys. Rev. A **52**, 4823 (1995).
[6] D. J. Santos and R. Loudon, Phys. Rev. A **52**, 1538 (1995).
[7] T. Gruner and D.-G. Welsch, Phys. Rev. A **53**, 1818 (1996).
[8] R. Matloob, Phys. Rev. A **69**, 052110 (2004).
[9] R. Matloob, Phys. Rev. A **70**, 022108 (2004).
[10] A. Tip, A. Moroz, and J. Combes, J. Phys. A **33**, 6223 (2000).
[11] A. Tip, L. Knöll, S. Scheel, and D. Welsch, Phys. Rev. A **63**, 043806 (2001).
[12] A. Tip, Phys. Rev. A **57**, 4818 (1998).
[13] H. T. Dung, L. Knöll, and D. Welsch, Phys. Rev. A **57**, 3931 (1998).
[14] S. Scheel, L. Knöll, and D. Welsch, Phys. Rev. A **58**, 700 (1998).
[15] J. R. Choi, J. Opt. B **5**, 409 (2003).
[16] M. Crocce, D. A. R. Dalvit, F. C. Lombardo, and F. D. Mazzitelli, Phys. Rev. A **70**, 033811 (2004).
[17] Z. Ying and G. Ben-Qing, Chin. Phys. Lett. **22**, 446 (2005).
[18] J. R. Choi, J. Phys. B **39**, 669 (2006).
[19] A. L. de Lima, A. Rosas, and I. A. Pedrosa, J. Phys. B **41**, 115503 (2008).
[20] E. Yablonovitch, Phys. Rev. Lett. **62**, 1742 (1989).
[21] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, Rev. Mod. Phys. **80**, 787 (2008).
[22] C. Yuce and Z. Ozcakmakli, J. Phys. A **42**, 035403 (2009).
[23] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media* (Butterworth-Heinemann, Oxford, 1998), 2nd ed.
[24] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), 2nd ed.
[25] H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. (N.Y.) **10**, 1458 (1969).
[26] H. R. Lewis, Phys. Rev. Lett. **18**, 510 (1967).
[27] A. L. de Lima, A. Rosas, and I. A. Pedrosa, Ann. Phys. (N.Y.) **323**, 2253 (2008).
[28] M. Abramovitz and I. A. Stegum, *Handbook of Mathematical Functions* (Dover, New York, USA, 1972).
[29] D. R. Smith, W. J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, Phys. Rev. Lett. **84**, 4184 (2000).
[30] P. M. Valanju, R. M. Walser, and A. P. Valanju, Phys. Rev. Lett. **88**, 187401 (2002).
[31] A. A. Houck, J. B. Brock, and I. L. Chuang, Phys. Rev. Lett. **90**, 137401 (2003).
[32] P. M. Alsing, J. P. Dowling, and G. J. Milburn, Phys. Rev. Lett. **94**, 220401 (2005).
[33] J. K. Kim and S. P. Kim, J. Phys. A **32**, 2711 (1999).
[34] I. A. Pedrosa, Hadronic J. **9**, 173 (1986).
[35] D. Stoler, Phys. Rev. D **1**, 3217 (1970).
[36] M. M. Nieto and L. M. Simmons, Jr., Phys. Rev. D **20**, 1321 (1979).
[37] B. J. Dalton, E. S. Guerra, and P. L. Knight, Phys. Rev. A **54**, 2292 (1996).