

Gravitational Wave Forms for Two- and Three-Body Gravitating Systems

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Different numbers of self-gravitating particles (in different types of periodic motion) are most likely to generate very different shapes of gravitational waves, some of which, however, can be accidentally almost the same. One such example is a binary and three-body system for Lagrange's solution. To track the evolution of these similar wave forms, we define a chirp mass to the triple system. Thereby, we show that the quadrupole wave forms cannot distinguish the sources. It is suggested that wave forms with higher ℓ th multipoles will be important for classification of them (with a conjecture of $\ell \leq N$ for N particles).

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Introduction.—Can one see an apple fall on a dark night? This is an inverse problem in gravitational wave astronomy. It can be specifically stated as “how can we know the source information such as the number of particles, their geometrical shape, and motion from observations of gravitational waves?” This problem is analogous to the well-known one for the sound, which was raised by Kac in Ref. [1]. Seeking an answer is beyond the scope of this Letter. As a specific issue which is related with the inverse problem, we shall examine gravitational radiation by a certain class of orbital motion of self-gravitating objects.

In the near future, direct detection of gravitational ripples (and consequently gravitational waves astronomy) will come true owing to a lot of effort by ongoing or designed detectors [2–7]. One of the most promising astrophysical sources is inspiraling and finally merging binary compact stars. Numerical relativity has succeeded in simulating merging neutron stars and black holes [8–12]. Analytic methods also have nicely prepared accurate wave form templates for inspiraling compact binaries, notably by the post-Newtonian approach (see [13,14] for reviews) and also by the black hole perturbations especially at the linear order in mass ratio (see also [15] for reviews). Bridges between the inspiraling stage and the final merging phase are currently under construction (e.g., [16,17]).

There is a growing interest in potential astrophysical sources of gravitational waves involving 3-body interactions (e.g., [18,19] and references therein). It is well known that even the classical three-body (or N -body) problem in Newtonian gravity admits an increasing number of solutions [20,21]. Some of the orbits are regular, while the others are chaotic. For simplicity, we focus on several periodic orbits of the three-body system: Lagrange's triangle, Henon's crisscross and Moore's figure eight, which are explained later (see also Fig. 1). Here, it should be noted that Nakamura and Oohara [22] studied numerically the luminosity of gravitational radiation by N test particles orbiting around a Schwarzschild black hole, as an extension of Detweiler's analysis of the $N = 1$ case [23] by using the Teukolsky equation [24], in order to show the phase cancellation effect, which had been pointed out by Nakamura

and Sasaki [25]. Their N particles are *test* masses but not self-gravitating. Another inverse problem of reconstructing the gravitational wave signal from the noisy data acquired by a network of detectors has been discussed (e.g., [26,27]). Our aim and setting are completely different from those of the existing works.

The purpose of this Letter is (1) to point out a case where very similar shapes of waves are generated accidentally by different numbers of particles and (2) to show that the usage of higher multipole contributions will be necessary for distinguishing such sources. In order to track the evolution of the wave forms, we shall define the chirp mass so as to extend to a three-body system. Thereby, we shall show that the octupole order is required to disentangle such very similar wave forms that coincide with each other at the quadrupole level. This will suggest that theoretical wave forms including sufficiently higher ℓ th order multi-

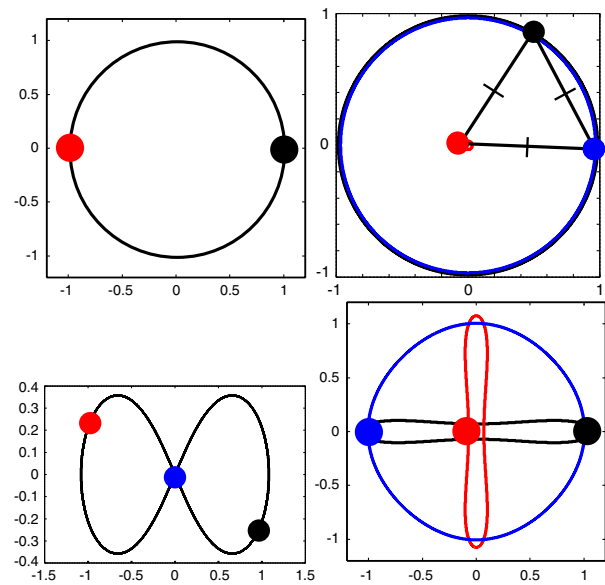


FIG. 1 (color online). Orbital shapes. (a) Top left: circular orbit for two-body system as a reference. (b) Top right: triangle solution by Lagrange. (c) Bottom right: crisscross orbit by Henon. (d) Bottom left: figure eight trajectory by Moore.

pole will be important for classification of sources generating such similar wave forms (with a conjecture about ℓ and N).

Throughout this Letter, we take the units of $G = c = 1$.

Some periodic orbits for three-body systems.—For simplicity, we assume that the motion of massive bodies follows the Newtonian equation of motion. It is impossible to describe all the solutions to the three-body problem even for the $1/r$ potential, as mentioned above. The simplest periodic solutions for this problem were discovered by Euler (1765) and by Lagrange (1772). The Euler's solution is a collinear solution in which the masses are collinear at every instant with the same ratios of their distances. The Lagrange's one is an equilateral triangle solution in which each mass moves in an ellipse in such a way that the triangle formed by the three bodies revolves. Let us take as another interesting solution the so-called crisscross orbit found by Henon in 1976 [28]. (See also [29] for the initial condition for each mass and its recent extensions.)

Since the figure eight solution was found first by Moore by topological classification [30], choreographic solutions have recently attracted increasing interests in astronomy, mathematics and physics, where a solution is called choreographic if every massive particles move periodically in a single closed orbit. The figure eight solution is that three bodies move periodically in a single figure eight [30]. The existence of such a figure eight orbit was proven by Chenciner and Montgomery [31], where the numerical initial condition for each mass is also given. This odd solution is remarkably stable in Newtonian gravity [32,33]. Heggie discussed a formation mechanism as an outcome from scattering of two binaries [34]. Its unicity up to scaling and rotation has been recently proven [35]. The trick figure eight remains true even if we consider the general relativistic (GR) effects at the post-Newtonian order [36] and also at the second post-Newtonian one [19]. This is a marked contrast to a binary case, which produces a complicated flowerlike pattern by the periastron advance in Einstein gravity. It is interesting to investigate relativistic effects on various kinds of orbital motions, which are discussed mostly in Newtonian gravity. It is a topic of future study. The radiation by the figure eight has been also investigated [18].

Gravitational waves.—In the previous part, we have mentioned several periodic solutions. Figure 2 shows the gravitational radiation by massive particles in these periodic motions, where the quadrupole formula is used.

Interestingly, the wave forms from a binary in circular motion and a three-body system constituting the Lagrange solution are the same in shape. It is worthwhile to mention that, if the third mass is extremely small, its contribution to the quadrupole waves becomes linear but not cubic in mass because its orbital radius is of the order of a triangle's side length, namely, bounded from above. If one adjusts properly distance r from an observer to the source with the same orbital period, the wave forms (including the amplitudes) could perfectly agree with each other.

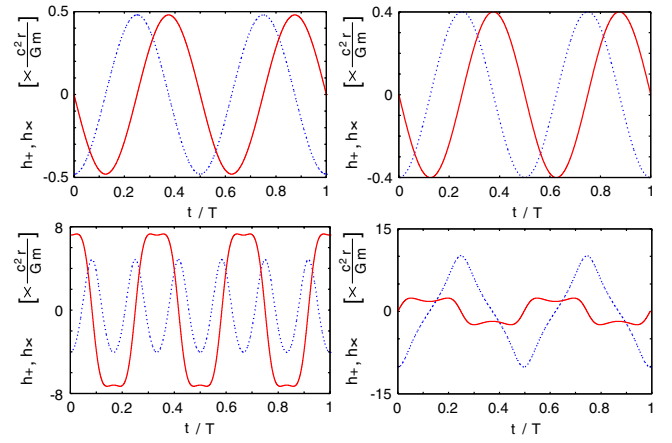


FIG. 2 (color online). Gravitational wave forms in arbitrary units ($T =$ orbital period). Dotted blue and solid red curves denote $+$ and \times modes, respectively. (a) Top left: Gravitational wave forms by binary system with a mass ratio of 2:3 in circular motion. (b) Top right: Lagrange's triangle solution for a mass ratio of 1:2:3. (c) Bottom right: Henon's crisscross. (d) Bottom left: Moore's figure eight. Crisscross and figure eight have larger curvatures in the orbital shapes than Keplerian and Lagrangian orbits, which lead to larger acceleration of the particles and thus relatively stronger radiation.

Chirp mass for three-body systems.—The wave forms shown above are valid only in the short term. The gravitational waves will gradually carry away the system's energy and angular momentum, and will eventually shrink the orbital size. Consequently, the amplitude and frequency of the waves will become larger and higher, respectively, with time. For a binary case, the frequency sweep is characterized by its chirp mass.

Here, we investigate the evolution of the wave forms for a three-body system for the Lagrange's solution (on x - y plane). The initial positions of each mass denoted by m_p ($p = 1, 2, 3$) are expressed as $\mathbf{x}_1 = (0, 0)$, $\mathbf{x}_2 = a(\sqrt{3}/2, 1/2)$, and $\mathbf{x}_3 = a(0, 1)$, where the side of a regular triangle is denoted as a . We take the coordinates such that the center of mass (COM) is at rest as $(x_{\text{COM}}, y_{\text{COM}}) = a(\sqrt{3}\nu_2/2, (\nu_2 + \nu_3)/2)$, where the total mass and mass ratio are denoted as $m_{\text{tot}} \equiv \sum_p m_p$ and $\nu_p \equiv m_p/m_{\text{tot}}$, respectively. The orbital frequency ω for the triangle satisfies $\omega^2 = m_{\text{tot}}/a^3$.

Henceforth, it is convenient to employ the COM coordinates (X, Y) that can be obtained by a translation from (x, y) . In the COM coordinates, the location of each mass at any time is expressed as $\mathbf{X}_p = a_p(\cos(\omega t + \theta_p), \sin(\omega t + \theta_p))$, where a_p is defined as $a_1 = \sqrt{x_{\text{COM}}^2 + y_{\text{COM}}^2}$, $a_2 = \sqrt{(3^{1/2}a/2 - x_{\text{COM}})^2 + (a/2 - y_{\text{COM}})^2}$, and $a_3 = \sqrt{x_{\text{COM}}^2 + (a - y_{\text{COM}})^2}$, respectively, and θ_p denotes the angle between the new X axis and the direction of each mass at $t = 0$ (see Fig. 3).

By using the standard quadrupole formula, the energy loss rate for the Lagrange's orbit is expressed as

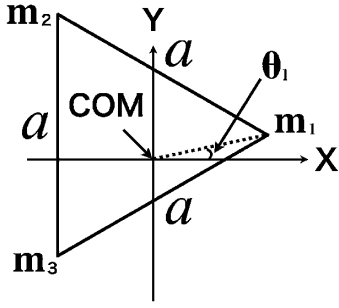


FIG. 3. Definition of θ_p in the Lagrange's triangle solution. The angle θ_p is measured from X axis to the direction of each mass at the initial time.

$$\frac{dE}{dt} = \frac{32}{5} m_{\text{tot}}^2 \omega^6 \left[\left(\sum_{p=1}^3 \nu_p a_p^2 \right)^2 - 4 \sum_{p < q} \nu_p \nu_q a_p^2 a_q^2 \sin^2(\theta_p - \theta_q) \right]. \quad (1)$$

The equation of motion for each body is rewritten in an effective one-body form as [20] $d^2 \mathbf{X}_p / dt^2 = -M_p \mathbf{X}_p / |\mathbf{X}_p|^3$, where we define the *effective mass* as

$$M_p = m_{\text{tot}} \left(\sum_{q \neq p} \nu_q^2 + \sum_{q, r \neq p} \nu_q \nu_r / 2 \right)^{3/2}. \quad (2)$$

The orbital frequency is the same for each body, which provides an identity as $M_p / a_p^3 = \omega^2$ from the above effective one-body equation of motion. One can reexpress a_p as $a_p = (M_p / m_{\text{tot}})^{1/3} a$ in terms of M_p because $\omega^2 = m_{\text{tot}} / a^3$.

For the triangle solution, we obtain the sum of the Newtonian kinetic and potential energy as

$$E_{\text{tot}} = -\frac{m_{\text{tot}}^2}{2a} \left[\sum_{p \neq q} \nu_p \nu_q - \sum_p \nu_p \left(\frac{M_p}{m_{\text{tot}}} \right)^{2/3} \right]. \quad (3)$$

By assuming adiabatic changes, we use the energy balance between the system energy loss and gravitational radiation. We find

$$\frac{1}{a} \frac{da}{dt} = -\frac{64}{5} \frac{m_{\text{tot}}^3}{a^4} \frac{\left\{ \sum_p \nu_p \left(\frac{M_p}{m_{\text{tot}}} \right)^{2/3} \right\}^2 - 2 \sum_{p \neq q} \nu_p \nu_q \left(\frac{M_p}{m_{\text{tot}}} \right)^{2/3} \left(\frac{M_q}{m_{\text{tot}}} \right)^{2/3} \sin^2(\theta_p - \theta_q)}{\sum_{p \neq q} \nu_p \nu_q - \sum_p \nu_p \left(\frac{M_p}{m_{\text{tot}}} \right)^{2/3}}, \quad (4)$$

which provides the shrinking rate of the triangle due to gravitational radiation reaction.

Since the gravitational waves frequency f_{GW} is twice of the orbital one, we have $f_{\text{GW}}^2 = m_{\text{tot}} / \pi^2 a^3$. Therefore, $d \ln f_{\text{GW}} / dt = -(3/2) d \ln a / dt$. Using this in Eq. (4), we obtain

$$\frac{1}{f_{\text{GW}}} \frac{df_{\text{GW}}}{dt} = \frac{96}{5} \pi^{8/3} M_{\text{chirp}}^{5/3} f_{\text{GW}}^{8/3}, \quad (5)$$

where we defined a chirp mass as

$$M_{\text{chirp}} = m_{\text{tot}} \left[\frac{\left\{ \sum_p \nu_p \left(\frac{M_p}{m_{\text{tot}}} \right)^{2/3} \right\}^2 - 2 \sum_{p \neq q} \nu_p \nu_q \left(\frac{M_p}{m_{\text{tot}}} \right)^{2/3} \left(\frac{M_q}{m_{\text{tot}}} \right)^{2/3} \sin^2(\theta_p - \theta_q)}{\sum_{p \neq q} \nu_p \nu_q - \sum_p \nu_p \left(\frac{M_p}{m_{\text{tot}}} \right)^{2/3}} \right]^{3/5}. \quad (6)$$

It is worthwhile to mention that the frequency sweep for the triple system can take the same form as that for binaries. One can show that Eq. (6) recovers the binary chirp mass in the limit of $m_3 \rightarrow 0$.

Equation (5) suggests that we cannot distinguish two cases of the binary and triple systems by using only the quadrupolar parts even if the frequency sweep is observed.

Octupole wave forms.—In a wave zone, the gravitational waves denoted by h_{ij}^{TT} can be expressed asymptotically in multipolar expansions [37]. The ratio of the octupole part to the quadrupole one is of the order of v/c , where v is a typical velocity of the matter. For instance, it is about 10% if $a = 100 m_{\text{tot}}$, which is assumed in order to exaggerate the octupole correction in Fig. 4.

After straightforward calculations, one can obtain an expression of octupolar parts of the gravitational waves

that are generated by the three-body system for the Lagrange's solution with arbitrary mass ratio. For instance, one of the relevant octupole moments is expressed as

$$I_{xxy} = \frac{1}{20} \sum_{p=1}^3 m_p |\mathbf{X}_p|^3 \sin(\omega t + \theta_p) - \frac{1}{4} \times \sum_{p=1}^3 m_p |\mathbf{X}_p|^3 \cos 3(\omega t + \theta_p). \quad (7)$$

I_{xyy} can be obtained by interchanges as $x \leftrightarrow y$ and $\sin \leftrightarrow \cos$. By using such analytic expressions, one can obtain the octupole contributions to wave forms.

It should be noted that no octupole radiation is emitted along the orbital axis for any planar motions. Let us take the observational direction along x axis. Then, we have only $+$ mode without \times mode. Figure 4 shows that a

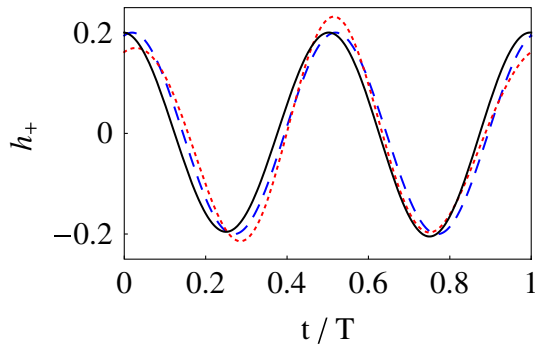


FIG. 4 (color online). Gravitational wave forms in arbitrary units for a binary (solid black curve) with $m_1:m_2 = 2:3$ and a Lagrange solution (dotted red curve) with $m_1:m_2:m_3 = 1:2:3$, where both the quadrupole and octupole parts are included. As a reference, we give the quadrupolar wave forms from the same sources (dashed blue curve). We assume $a = 100m_{\text{tot}}$ in order to exaggerate a correction by the octupole (nearly 10% expected in this figure). One can see that the dashed blue curve will overlap with the solid black one after they are shifted by choosing the initial phase. This coincidence is because the octupolar waves for the binary case are proportional to the mass difference [38] and thus relatively small in this figure.

difference between the wave forms (one by the binary and the other by the triplet) comes up at the octupole order. The octupole radiation amplitude by binaries is proportional to the mass difference [38]. On the other hand, the octupole radiation exists for triangles even if they are all equal masses. Cases of various mass ratios and observational directions are a topic of future study.

Conclusion.—In summary, we have examined different numbers of self-gravitating particles in gravitational waves astronomy. In order to track the evolution of the similar wave forms from the two-body and three-body systems, we have defined a chirp mass to the three-body case. We have shown that the wave forms at the quadrupole level cannot distinguish the sources even with observing frequency sweep. Our example suggests that theoretical wave forms including higher multipole parts will be important for classification of such similar imprints. Higher post-Newtonian corrections both to the wave forms and to the motion of bodies should be incorporated. This is a topic of future study. In particular, the stability of the Lagrange orbit due to GR effects is poorly understood.

It is conjectured by induction from our result that classification of N (or fewer) particles producing (nearly) the same wave forms requires inclusions of the ℓ th multipole part with $\ell \leq N$. Cases of $\ell < N$ are realized for instance by the crisscross and figure eight. Proving (or disproving) this conjecture is left as future work.

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