Effective Medium Theory for Reaction Rates and Diffusion Coefficients of Heterogeneous Systems

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A simple effective medium theory is derived for spatially heterogeneous nonlinear reaction-diffusion media. Its validity is tested through comparisons with simulations of front and pulse propagation in systems with spatially varying diffusion coefficients and reaction rates. The theory is able to predict wave speeds if the characteristic front width is much larger than the length scale of the heterogeneities. This condition is violated in media with isolated or weakly connected sites. However, the theory nevertheless provides good results in cases where it correctly predicts the percolation threshold of the medium.

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Spatial structures often occur in chemical [1] and biological [2] systems and determine many of their properties. Reaction-diffusion (RD) models for such pattern formation usually assume that the system is spatially homogeneous, but experiments on the Belousov-Zhabotinsky reaction (BZR) [3,4], catalysis [5], and cardiac tissue [6,7] have shown that spatial heterogeneity can strongly affect spatial structure and dynamics. For example, BZR experiments on microemulsions have demonstrated that the structural composition of the emulsion can be varied to tune the concentration patterns in this system [8]. Heterogeneities are intrinsic in biological systems, and spatially heterogeneous RD equations have been employed to model processes such as intracellular calcium waves [9] and electrical propagation in the heart [10]. A large number of studies, however, assume homogeneous RD equations to model biological and chemical systems, but a general and simple framework to relate both types of approaches is still missing.

Frequently, heterogeneities may be distributed in a complicated or even random manner specific to a particular system. For instance, each piece of excitable cardiac tissue has a different heterogeneous distribution of excitable cells with different properties. Rather than relying on models that assume a specific set of inhomogeneities, it is far better to construct models that incorporate only generic features of the heterogeneity. If the spatial scale of the heterogeneity is much smaller than the scale of the RD pattern, a description based on RD equations for an effective homogeneous medium should be able to describe the main pattern features. In this spirit, homogenization procedures based on explicit analytical [11,12] or implicit numerical [13] averaging methods have been employed to obtain wave properties in (mostly periodic) one-dimensional heterogeneous RD media. In two and three dimensions, homogenization results were obtained for selected important biophysical problems such as propagation in cardiac tissue [14] and calcium dynamics in cardiac myocytes [15]. These results depend on specific details of the model. Here we present a simple method to obtain effective parameters for random heterogeneous RD systems and test its validity and limitations by direct simulations in various heterogeneous model systems.

Since pattern formation relies on the interplay between nonlinear kinetics and diffusion, we extend effective medium theories for linear kinetics [16] to obtain effective reaction rates and diffusion coefficients for nonlinear RD systems. To test the effective medium theory, pattern dynamics in models of two-dimensional random binary bistable and excitable media with small-scale heterogeneities in diffusion and reaction were studied. In random binary media, the diffusion coefficient depends on space and takes only two different values: $D_2 < 1$ with a probability ϕ and $D_1 = 1$ with probability $1 - \phi$. Simulations are in excellent agreement with the analytical predictions of effective medium theory provided the characteristic pattern length is larger than the length scale of the heterogeneity (homogenization condition), i.e., when D_2 has a sufficiently large value. In the limiting case $D_2 \to 0$, our random media consist of a conducting phase 1 and an isolating phase 2. Macroscopic transport and waves are observed only if the fraction of phase 1 is large enough for percolation. Although the homogenization condition is violated for $D_2 \to 0$, the effective medium theory predicts correctly the dynamics if it reproduces the percolation threshold correctly. The results of this study provide a basis and criteria for the construction of effective models for heterogeneous media.

We consider a heterogeneous medium composed of a random distribution of two types of domains, where chemical species react and diffuse. Particles can diffuse throughout the system, but their diffusion coefficients and reaction rates may differ in the different domains and have the values D_1, R_1 and D_2, R_2 in type-1 and type-2 domains, respectively. The dynamics in the heterogeneous medium is described by the RD equation

$$\partial_t c(r,t) = \nabla \cdot [D(r)\nabla c(r,t)] + R(c(r,t), r), \tag{1}$$

where c and R are vectors of the concentrations and reaction rate fields for a multicomponent system. The spatially dependent terms are defined so that $D(r) = D_i$, $R(c(r,t), r) = R_i(c(r,t))$ in domain types $i = 1, 2$.

If the characteristic domain size ℓ_{het} is small compared to the characteristic pattern scale ℓ_p, one may describe the RD dynamics in terms of effective medium equations obtained by coarse-graining over distances that are large compared to ℓ_{het}. Letting the angular bracket $\langle \ldots \rangle$ represent such a coarse-graining, we may replace the averaged equation

$$\partial_t \langle c(r,t)\rangle = \langle \nabla \cdot [D(r)\nabla c(r,t)]\rangle + \langle R(c(r,t), r)\rangle \tag{2}$$

by an effective medium RD equation

$$\partial_t \langle c(r,t)\rangle = D_e \nabla^2 \langle c(r,t)\rangle + R_e[\langle c(r,t)\rangle], \tag{3}$$

with an effective diffusion coefficient D_e and an effective reaction rate $R_e[\langle c(r,t)\rangle]$. Given a random distribution of the two phases and a volume fraction ϕ, the effective diffusion coefficient is given by an implicit relation [16] that depends on D_1, D_2, and ϕ:

$$(1-\phi)\frac{D_1 - D_e}{D_1 + (d-1)D_e} + \phi \frac{D_2 - D_e}{D_2 + (d-1)D_e} = 0, \tag{4}$$

where d is the spatial dimension of the system. Equivalent results for the diffusion coefficient exist for the conductivity of mixtures of isotropic materials [17], for spherical inclusions in a conducting material [18], and for inhomogeneous transport in resistor networks [19]. Equation (4) predicts a percolation threshold of $\phi^* = 1/2$ in $d = 2$ because the effective diffusion coefficient for $D_2 = 0$ is $D_e = D_1(1 - 2\phi)$.

A simple expression for the effective nonlinear reaction rate can be obtained if the characteristic domain size ℓ_{het} is much smaller than the distance ℓ_D that particles diffuse in the mean reaction time τ_R: $\ell_{het}/\ell_D \ll 1$, where $\ell_D^2 = D_e \tau_R$. In this case reactions will occur homogeneously within the two phases, and spatial inhomogeneity arises on scales longer than the average domain size. Under these conditions we have

$$R_e = (1-\phi)R_1[\langle c(r,t)\rangle] + \phi R_2[\langle c(r,t)\rangle]. \tag{5}$$

We consider a one-component bistable medium with reaction rate [20] $R(c, r) = k(r)c(1-c)(c-a)$ and diffusion coefficient $D(r)$. The spatially homogeneous system supports a front that propagates with a stationary shape and constant velocity:

$$V_0 = \sqrt{Dk/2}(1 - 2a). \tag{6}$$

The characteristic pattern scale ℓ_p is given by the interface width $\ell_p = 4\sqrt{2D/k}$. For our choice of parameters, $\ell_p = 0.8$ for $D = 1$. The heterogeneity scale is set to be $\ell_{het} = 0.1$, so that the condition for homogenization $\ell_{het} \ll \ell_p$ is clearly violated, once $D < 0.2$. If we spatially discretize the heterogeneous medium into a grid of points connected by links, different cases can be studied. (i) A link between two neighboring sites has either normal diffusion (D_1) or a reduced diffusion coefficient (D_2) corresponding to a weak link. The probability to find such weak links is ϕ_{wl}, and there is no correlation between links. (ii) We consider the system with weak links and introduce a probability ϕ_{nr} for a site to be nonreactive ($k_2 = 0$). We assume that both types of heterogeneities appear with the same probability $\phi_{nr} = \phi_{wl}$ and that their locations are uncorrelated. (iii) Defects in the system are studied. A defect is a site, where all links have a reduced diffusion (D_2). They appear with a probability ϕ_d. (iv) Nonreactive ($k_2 = 0$) defects can be also considered. We consider such heterogeneities because cases (i) and (iii) lead for $D_2 = 0$ to bond and site percolation, respectively. Cases (ii) and (iv) are extensions of cases (i) and (iii) with heterogeneous reaction.

In the numerical simulations, a front was initiated on one side of a 2D system and allowed to propagate through a homogeneous medium until a stationary shape was achieved. After the front entered an inhomogeneous region, the velocity of the front was measured. The wave patterns are qualitatively similar for all cases. Figure 1 shows two examples for case (i). Figure 1(a) corresponds to an intermediate fraction of type-2 domains. In Fig. 1(b), the fraction of weak links is close to the percolation threshold ($\phi_{wl}^* = 0.5$).

We carried out a systematic study of the dependence of the wave velocity on ϕ and D_2. The explicit dependence of the effective diffusion coefficient on ϕ_{wl} (probability of weak links) and D_2 can be obtained from Eq. (4), and the dependence of the effective reaction rate on ϕ_{nr} (probability of nonreactive sites) from Eq. (5): $k_e = (1 - \phi_{nr})k$. Using these results in Eq. (6), we can compare the simu-

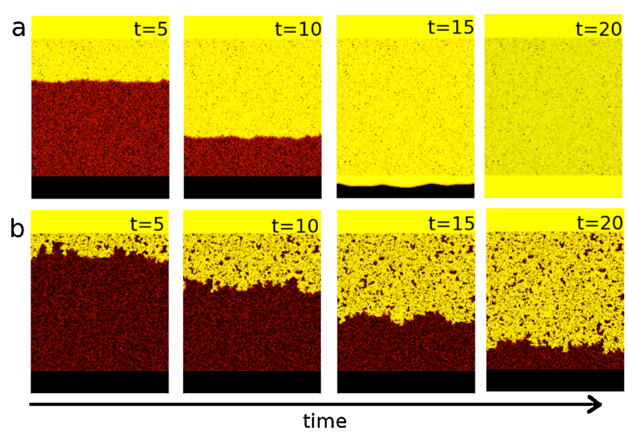

FIG. 1 (color online). Front evolution in systems with weak links ($D_2 = 0$) with densities (a) $\phi_{wl} = 0.3$ and (b) $\phi_{wl} = 0.47$. Fronts (bright areas) evolve from top to bottom. Different levels of darkness represent heterogeneities, which extend in 5 s.u. $< x <$ 35 s.u. (size 30 × 40 s.u.^{2}). Periodic boundary conditions in the horizontal direction are applied.

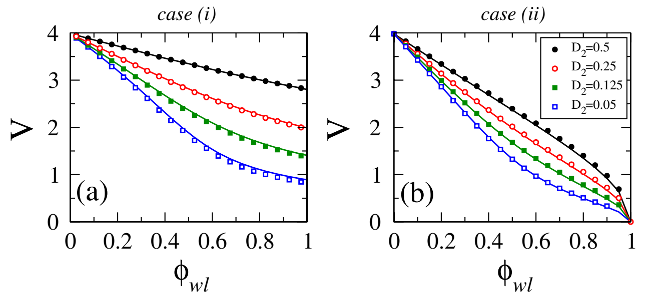

FIG. 2 (color online). Dependence of front velocity on D_2 and $\phi_{\rm wl}$: (a) weak links and (b) weak links and nonreactive sites ($\phi_{\rm nr} = \phi_{\rm wl}$). Points, average over 100 realizations; solid lines, theory. The legend applies to both figures.

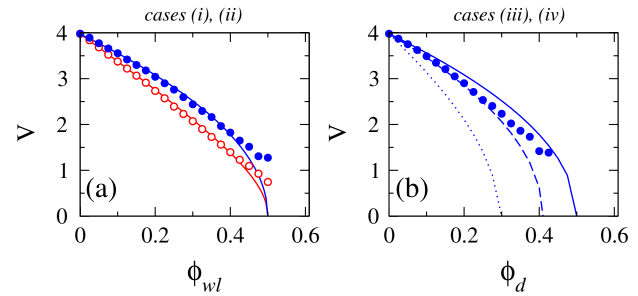

FIG. 3 (color online). (a) Dependence of front velocity on the probability $\phi_{\rm wl}$ of zero-diffusion links ($D_2 = 0$) with reactive (dark circles) and with nonreactive (open circles) sites. (b) Dependence of front velocity on the probability ϕ_d of zero-diffusion defects ($D_2 = 0$). Points, average over 100 realizations; solid lines, original theory; dashed line, modified formula; dotted line, modified theory.

lation results with the predictions of the theory:

$$V = V_0\sqrt{D_e k_e / D k}. \tag{7}$$

There are no qualitative differences between the behaviors in 2D and 3D; thus, we present only results for 2D here.

For case (i) the reactivity of the medium is homogeneous. Figure 2(a) shows that the effective medium theory is able to reproduce correctly the results of the simulations. In addition, we consider the combination of weak links with nonreactive sites [case (ii)]. For $\phi_{\rm nr}$ close to 1, the front does not propagate and the velocity goes to zero. The simulation results [Fig. 2(b)] are well reproduced by Eq. (7). Figure 2 shows that the effective medium theory can be applied to the propagation of fronts through a medium where the diffusivity of some links has been reduced. There is no fitting parameter in the theory. The theory can be applied for sufficient large D_2. It works even in the limit of $D_2 = 0$ if the theory correctly predicts the percolation threshold. The dependence of the velocity is $V = V_0\sqrt{1 - d\phi_{\rm wl}/(d-1)}$ for case (i) and $V = V_0\sqrt{1 - d\phi_{\rm wl}/(d-1)}\sqrt{1-\phi_{\rm nr}}$ for case (ii) [see Fig. 3(a)]. We observe in Fig. 3 that the effective medium theory can be applied for small values of $\phi_{\rm wl}$ but fails for values of $\phi_{\rm wl}$ close to the percolation threshold. In this case, larger clusters form and homogenization fails.

The percolation threshold is different for zero-diffusion links and zero-diffusion defects corresponding to the cases of bond and site percolation, respectively. The theory predicts a percolation threshold which is correct for bond percolation and allows quantitative comparisons [Fig. 3(a)]. However, the prediction in 2D of the percolation threshold ($\phi^* = 0.5$) differs substantially from the site percolation threshold ($\phi_d^* = 0.41$). Using the modified formula $V = V_0\sqrt{1 - \phi/\phi_d^*}$, we can reproduce the numerical results [dashed line in Fig. 3(b)] for $D_2 = 0$ better than the original theory [solid line in Fig. 3(b)].

While Eq. (4) applies to weak links, it is not suitable to describe defects. In order to find an improved expression, we transform the probability ϕ_d that a site is a defect to a different probability ϕ_{dl} that a link belongs to a defect and, therefore, is assigned a weaker diffusion coefficient. If ϕ_d is the fraction of defects and $1 - \phi_d$ the one of normal nodes, the probability of a normal link would be $(1-\phi_d)^2$ and the probability of a weak link is $\phi_{dl} = 2\phi_d - \phi_d^2$. We replace the probability ϕ_d by the new probability ϕ_{dl} in Eq. (4) but keep $\phi_{\rm nr} = \phi_d$ in the expression for k_e for case (iv). For comparison with simulations, see Fig. 4. The modified effective medium theory is able to reproduce the numerical results for large values of D_2 but exhibits discrepancies for small values of D_2 (Fig. 4), in particular, for $D_2 = 0$ [dotted line in Fig. 3(b)], due to the incorrect prediction of the percolation threshold.

In summary, the effective medium theory works for large enough values of D_2. While theory and simulation agree for bond percolation, this is not the case for site percolation. The introduction of a modified percolation threshold describes 2D site percolation and the limit $D_2 \to 0$ well. However, the theory fails for defects. The introduction of a new fraction ϕ_{dl} for the links leads to good agreement for large and intermediate values of D_2.

The addition of a second equation for an inhibitor [21] leads to the production of pulses instead of fronts and may

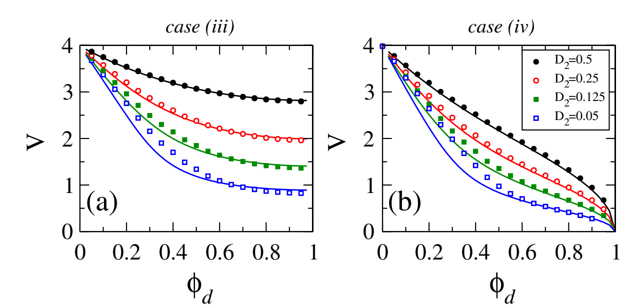

FIG. 4 (color online). Dependence of the velocity of a front on D_2 and ϕ_d: (a) reactive defects and (b) nonreactive defects. Points, average over 100 realizations; solid lines, modified theory. The legend applies to both figures.

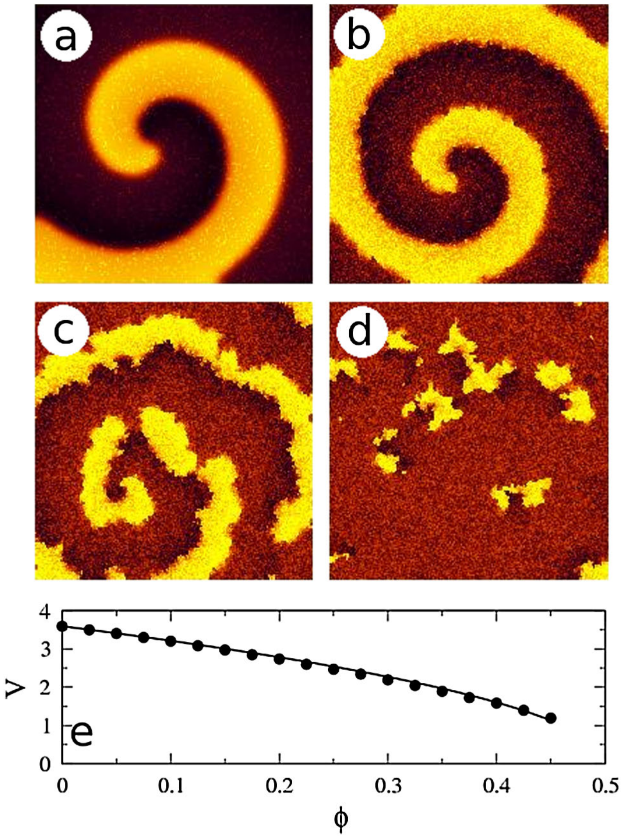

FIG. 5 (color online). Spiral waves in heterogeneous excitable media. As the time increases, the density of weak links ($D_2 = 0.01$) is increased. Snapshots for $\phi_{\rm wl} = 0.025$, $\phi_{\rm wl} = 0.275$, $\phi_{\rm wl} = 0.475$, and $\phi_{\rm wl} = 0.500$ (size 20×20 s.u.^{2}). For the parameters, see [20,21].

convert the bistable medium into an excitable one. The width of pulses is affected by inhomogeneity: It becomes smaller, and finally wave breakup is observed. An example of spiral wave breakup is shown in Fig. 5. The pitch of the spiral and the size of the pulse decreases as $\phi_{\rm wl}$ increases until the spiral breaks in small pieces which are not able to propagate. A comparison of the pulse velocity with the prediction of the effective medium theory yields good agreement [Fig. 5(e)].

Cardiac tissue is an excitable medium where the characteristic size of the heterogeneities often is much smaller than the size of the pulse, and the results observed for fronts can be directly applied to large pulses. However, cell cultures display patterns like the ones observed in Fig. 5 [7,22]. There have been several attempts to construct homogenization theories motivated by cardiac tissue, e.g., assuming a periodic change in the diffusion coefficient [11]. Our results are independent of the specific choice of the RD equations and assume a random distribution of heterogeneities. The theory works also for spatiotemporal disorder provided its correlation time is longer than the diffusion time ($\ell_{\rm het}^2/D_2$).

In conclusion, we have demonstrated the wide applicability of effective medium theory to the propagation of pulses in RD systems with a heterogeneous distribution of links (bond percolation) or defects (site percolation).

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[20] Value of the parameters: $k = 50$, $D = 1$, and $a = 0.1$. Numerical parameters: $\Delta t = 0.002$ time units (t.u.) and $\Delta x = 0.1$ spatial units (s.u.).
[21] We consider the two equations $\partial_t c = R_c + \nabla \cdot [D(r)\nabla c]$ and $\partial_t w = R_w$, where $R_c = k[c(1-c)(c-a) - w]$ and $R_w = bc - w$, which gives rise to excitable dynamics for $b > 0.19$ and recovers the original model for $b = 0$. Here we employ $b = 0.2$.
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