Electromagnetic Wave Scattering by Schwarzschild Black Holes

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We analyze the scattering of a planar monochromatic electromagnetic wave incident upon a Schwarzschild black hole. We obtain accurate numerical results from the partial wave method for the electromagnetic scattering cross section and show that they are in excellent agreement with analytical approximations. The scattering of electromagnetic waves is compared with the scattering of scalar, spinor, and gravitational waves. We present a unified picture of the scattering of all massless fields for the first time.

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Black holes are thought to be efficient catalysts for the liberation of rest-mass energy. As such, black holes are implicated in the most energetic phenomena in the known universe (e.g., gamma ray bursts). On the other hand, after a turbulent youth, many black holes settle into a quiescent old age. Some estimates suggest there may be up to a billion quiescent stellar-mass black holes within our galaxy [1]. Their existence may be inferred from, for example, the transient lensing of background sources; a handful of events have so far been observed [2]. A possibility for future consideration is that quiescent black holes may be indirectly identified from the "fingerprint" they leave on radiation that impinges upon them.

Over the last four decades, some clues about the properties of any such "fingerprint" have been uncovered. For example, a time-dependent perturbation incident upon a black hole will excite characteristic *damped ringing* in response. The frequencies and decay rates of the ringing are linked to the well-studied quasinormal mode spectrum [3]. Black holes illuminated by long-lasting planar radiation will create interference patterns, and rotating black holes will create distinctive polarization patterns [4]. Both effects depend strongly on the ratio of horizon size to wavelength. Hence, it is conceivable that future gravitational-wave detectors may be able to identify the fingerprint from rapid and distinctive variations across a narrow frequency band. Nevertheless, inferring the presence of quiescent black holes from such clues must remain a challenge for future decades.

Scattering by black holes is of foundational interest in both black hole physics [5] and scattering theory [6]. Many authors have studied the simplest time-independent scenario, in which a black hole is subject to a long-lasting, monochromatic beam of radiation. Here, the key dimensionless quantity is the ratio r_h/λ, where r_h is the horizon size of the black hole, and λ is the wavelength of the incident wave. The interference pattern depends also on the spin s of the perturbing field, with $s = 0, 1/2, 1$, and 2 corresponding to scalar, neutrino, electromagnetic, and gravitational fields, respectively.

To the best of our knowledge, the first paper outlining a calculation of wave scattering cross section in the spacetime of a black hole was published by Matzner [7] in the late sixties. Since then, planar wave scattering from black holes has received much attention, especially in Schwarzschild and Kerr spacetimes (see Refs. [5,8,9] for comprehensive accounts on the subject). Let us briefly review a sample of the literature for the simplest case, the Schwarzschild black hole, for which the scattering of monochromatic fields of all spins ($s = 0, 1/2, 1$, and 2) has been studied through the years. The case of scalar waves ($s = 0$) was extensively studied by Sanchez [10,11], both analytically and numerically, and an accurate numerical study was later performed by Andersson [12]. Fermion ($s = 1/2$) scattering by a Schwarzschild black hole was the subject of a recent study [4], in which the authors also elucidated the effect of nonzero field mass. The case of electromagnetic waves ($s = 1$) was studied analytically by Mashoon [13] and Fabbri [14], and some results were obtained in the low- and high-frequency limits. Gravitational waves ($s = 2$) were the first to be studied in black hole scattering [15], and are the subject of old [5,16] and new [17,18] works.

In this Letter, we present the first detailed numerical investigation of the scattering of an electromagnetic plane wave by a Schwarzschild black hole. This work fills a gap in the literature and complements recent numerical studies of the scalar [19], fermionic [4], and gravitational [18] cases. We take this opportunity to present a unified picture of all four fields, for the first time.

We use natural units with $c = G = 1$ and the metric signature $(+ - - -)$.

The line element of Schwarzschild spacetime can be written as

$$ds^2 = f(r)dt^2 - [f(r)]^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where $f(r) = 1 - 2M/r$, with M being the black hole mass. The Schwarzschild solution describes static and chargeless black holes, with event horizon at $r_h = 2M$.

The Lagrangian density of the electromagnetic field in the modified Feynman gauge is [20]

$$\mathcal{L} = -\sqrt{-g}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}G^2\right]$$

with $g = \det(g_{\mu\nu})$, $G \equiv \nabla^\mu A_\mu + K^\mu A_\mu$, and $K^\mu = (0, df/dr, 0, 0)$. The equations of motion are found to be

$$\nabla_\nu F^{\nu\mu} + \nabla^\mu G - K^\mu G = 0. \tag{2}$$

The two physical polarizations in Schwarzschild spacetime can be written as

$$A_\mu^{(I\omega lm)} = \left(0, \frac{\varphi^I_{\omega l}(r)}{r^2}Y_{lm}, \frac{f}{l(l+1)} \times \frac{d}{dr}[\varphi^I_{\omega l}(r)]\partial_\theta Y_{lm}, \frac{f}{l(l+1)} \times \frac{d}{dr}[\varphi^I_{\omega l}(r)]\partial_\phi Y_{lm}\right)e^{-i\omega t}, \tag{3}$$

$$A_\mu^{(II\omega lm)} = (0, 0, \varphi^{II}_{\omega l}(r)Y^{lm}_\theta, \varphi^{II}_{\omega l}(r)Y^{lm}_\phi)e^{-i\omega t}, \tag{4}$$

with $\omega > 0$, and $l \geq 1$. [See, e.g., Ref. [20] for a discussion on the possible solutions of Eq. (2).] In Eqs. (3) and (4), Y_{lm} and Y^{lm}_a ($a = \theta$, ϕ) are the scalar and vector spherical harmonics [21], respectively, and $\varphi^\lambda_{\omega l}(r)$ satisfy the following equation

$$[\omega^2 - V(r)]\varphi^\lambda_{\omega l}(r) + f\frac{d}{dr}\left(f\frac{d}{dr}\varphi^\lambda_{\omega l}(r)\right) = 0, \tag{5}$$

with $\lambda = I, II$, where the effective potential is $V(r) = f[l(l+1)/r^2]$.

To evaluate the solutions of the Eq. (5) in the asymptotical limits, we use the Wheeler coordinate, defined as $x = r + r_h \ln(r/r_h - 1)$, and rewrite Eq. (5) as

$$(\omega^2 - V)\varphi^\lambda_{\omega l}(x) + \frac{d^2}{dx^2}[\varphi^\lambda_{\omega l}(x)] = 0. \tag{6}$$

For the computation of the scattering cross section we need only to consider modes incoming from the past null infinity $\mathcal{J}^-$. For these modes, the asymptotic solutions of Eq. (6) are [22]

$$\varphi^\lambda_{\omega l}(x) \approx A^\lambda_{\omega l}T^\lambda_{\omega l}e^{-i\omega x}, \tag{7}$$

for $x \to -\infty$ ($r \to r_h$) and

$$\frac{\varphi^\lambda_{\omega l}(x)}{\omega x} \approx A^\lambda_{\omega l}[(-i)^{l+1}h^{(1)*}_l(\omega x) + R^\lambda_{\omega l}i^{l+1}h^{(1)}_l(\omega x)], \tag{8}$$

for $x \gg r_h$ ($r \gg r_h$). Here, $h^{(1)}_l(x)$ denote the spherical Bessel functions of the third kind [23], and $|R^\lambda_{\omega l}|^2$ and $|T^\lambda_{\omega l}|^2$ are the reflection and transmission coefficients, respectively, which satisfy $|R^\lambda_{\omega l}|^2 + |T^\lambda_{\omega l}|^2 = 1$. $A^\lambda_{\omega l}$ is a normalization constant which is not important for the scattering properties.

The phase shifts are related to the reflection coefficient by

$$e^{2i\delta^\lambda_l(\omega)} = (-1)^{l+1}R^\lambda_{\omega l}. \tag{9}$$

For Schwarzschild black holes, the phase shifts of the two different physical polarizations are the same, i. e., $\delta^I_l(\omega) = \delta^{II}_l(\omega) = \delta_l(\omega)$ [14].

The differential electromagnetic scattering cross section is [24]

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\omega^2}\left|\sum_{l=1}^{\infty}\frac{2l+1}{l(l+1)}e^{+2i\delta_l(\omega)}\left[\frac{P^1_l(\cos\theta)}{\sin\theta} + \frac{d}{d\theta}P^1_l(\cos\theta)\right]\right|^2, \tag{10}$$

where $P^m_l(\cos\theta)$ are the associated Legendre functions. Note that Eq. (10) takes into account the contributions from the two physical polarizations, and it is valid for both linearly and circularly polarized waves. The polarization properties of the initial wave remain unchanged in the scattering by nonrotating black holes [13].

For small angles, this scattering cross section is the same for the massless scalar and electromagnetic fields, and it is given by [13,14]

$$\frac{d\sigma}{d\Omega} \approx \frac{16M^2}{\theta^4}. \tag{11}$$

In fact, the same behavior for small angles is obtained for massless fermionic and gravitational fields scattered in Schwarzschild spacetime [4,5,17].

The glory approximation for scattering of electromagnetic waves by a Schwarzschild black hole can be determined using the strong field approximation for the deflection angle (which was first obtained by Darwin [25]) together with the general glory formula [26], namely,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta\approx\pi} \approx 2\pi\omega b_g^2\left|\frac{db}{d\theta}\right|_{\theta=\pi}[J_{2s}(\omega b_g \sin\theta)]^2, \tag{12}$$

where b is the impact parameter of the incident particle, $J_l(x)$ are the Bessel functions of first kind, b_g is the impact parameter for which the scattering angle is π, and s is the particle spin. For the electromagnetic field ($s = 1$), the glory scattering cross section is given by

$$\frac{1}{M^2}\frac{d\sigma}{d\Omega}\bigg|_{\theta\approx\pi} \approx 30.75M\omega[J_2(5.36M\omega\sin\theta)]^2, \qquad (13)$$

where the coefficients (30.75 and 5.36) were obtained by solving the geodesic equation numerically [27].

In order to evaluate the electromagnetic scattering cross section numerically, we first solve Eq. (5) and match the solution with Eqs. (8) and (9). The scattering cross section for arbitrary frequencies and angles is obtained through Eq. (10). The numerical method used here is analogous to the one described in Ref. [28]. We have employed an iterative method similar to that used in Refs. [18,29] to improve the numerical convergence of the partial wave series.

In Fig. 1, we show the differential electromagnetic scattering cross section of Schwarzschild black holes computed numerically for different values of the incident wave frequency ($M\omega = 1, 2, 3, 4$). We also show the results for the glory scattering [given by Eq. (13)] in each case. Our numerical results are in excellent agreement with the glory approximation for $\theta \approx \pi$.

The zero in the backward direction (Fig. 1) is a consequence of the parallel-transport of the polarization vector along a geodesic. Consider an incoming geodesic ray in the z-direction which orbits the hole once, to return in the opposite direction ($\theta = \pi$). Assume, without loss of generality, it has an electric-field vector in the x direction. If the ray orbits in the x-z plane, then the vector will be reversed, whereas if the ray orbits in the y-z plane, the vector remains unchanged. Hence, by integrating over the circular degeneracy (all orbital planes), there is perfect cancellation. Similar arguments hold for other spins [5,30].

In Fig. 2, we plot the differential scattering cross section of Schwarzschild black holes for massless scalar ($s = 0$), massless spinor ($s = 1/2$), electromagnetic ($s = 1$), and gravitational ($s = 2$) waves. As expected, in the backward direction, all nonzero spin massless fields have vanishing cross section, whereas the zero-spin (scalar) massless field has a glory maximum at $\theta = \pi$. We see that scalar ($s = 0$) and electromagnetic ($s = 1$) scattering cross sections are very similar in the angular region $45° < \theta < 160°$. All integer spin fields ($s = 0, 1, 2$) behave similarly for $45° < \theta < 120°$. Bosonic ($s = 0, 1, 2$) and fermionic ($s = 1/2$) scattering cross sections oscillate in antiphase throughout almost all the angular range of Fig. 2 (except near $\theta \sim \pi$).

The regular oscillations in the cross sections of Fig. 2 can be understood semiclassically. They arise from the interference of rays passing in opposite senses around the hole. There is a "path difference" between the rays passing through angles θ and $2\pi - \theta$. A maximum (minimum) occurs when the path difference is an integer (half-integer) multiple of the wavelength λ. Hence, the angular width of the oscillations is inversely proportional to $M\omega$.

In summary, we have studied the scattering of a monochromatic planar electromagnetic wave by a

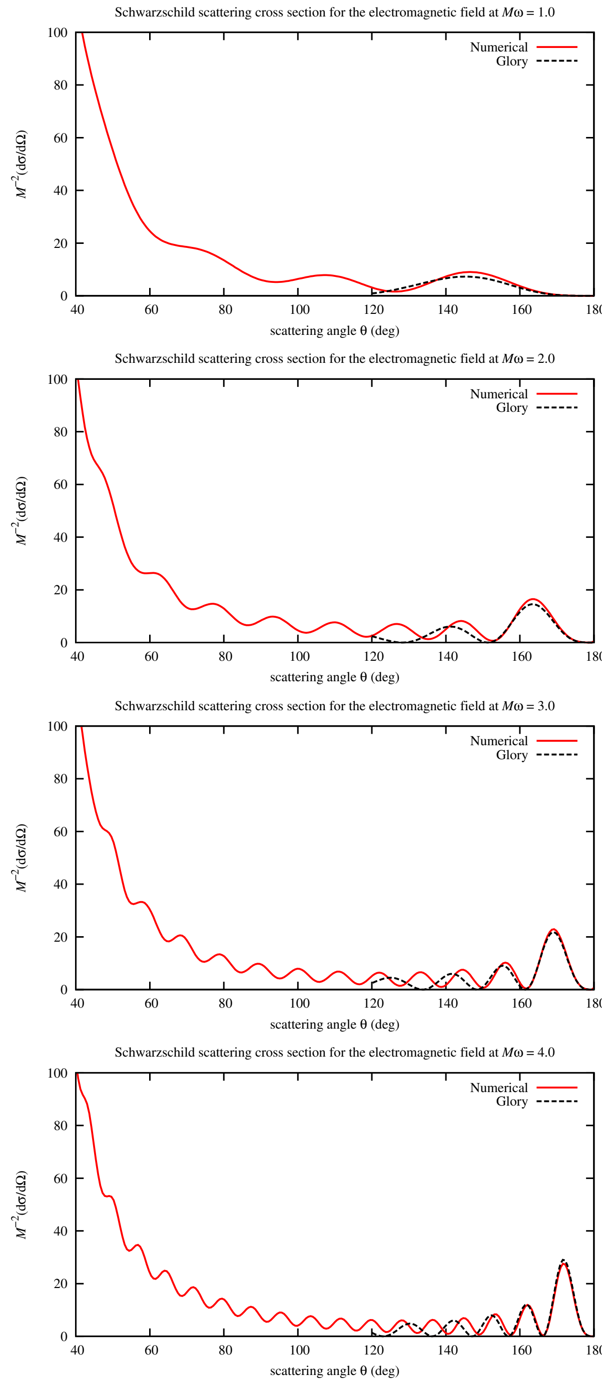

FIG. 1 (color online). Electromagnetic scattering cross section of Schwarzschild black holes for different choices of $M\omega$. We compare our numerical results (solid lines) with the glory approximation (dashed lines) given by Eq. (13), obtaining excellent agreement for $\theta \approx \pi$.

Schwarzschild black hole. We have applied the partial wave method to obtain the differential scattering cross section numerically, for different values of the frequency of the incident plane wave and for different values of the scattering angle. We have presented graphs with accurate numerical results for massless fields of all spins, that is,

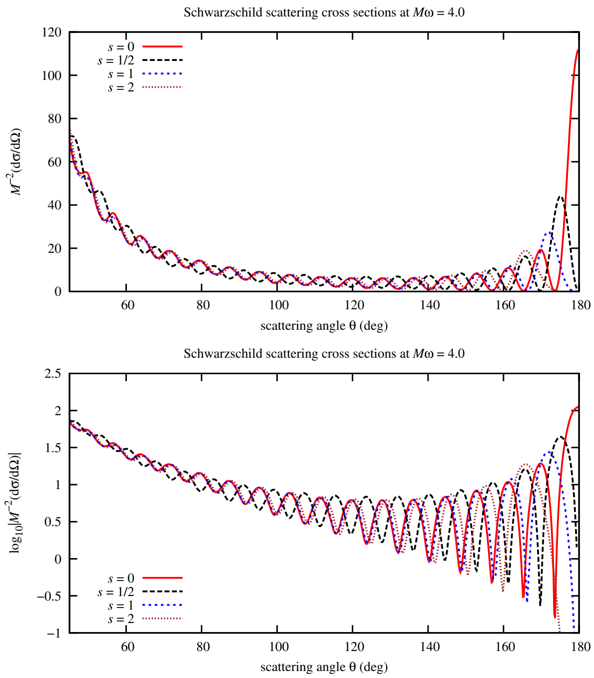

FIG. 2 (color online). Scattering cross section of Schwarzschild black holes for massless scalar ($s = 0$), electromagnetic ($s = 1$), gravitational ($s = 2$), and massless fermionic fields ($s = 1/2$) at $M\omega = 4.0$. Note the log scale on the vertical axis of the lower plot. We see that, as all other nonzero spin fields, the electromagnetic wave has a vanishing scattering cross section in the backward direction.

scalar ($s = 0$), fermionic ($s = 1/2$), electromagnetic ($s = 1$), and gravitational ($s = 2$) fields. All nonzero spin massless fields have a vanishing scattering cross section in the backward direction ($\theta = \pi$), whereas the scattering cross section of the massless scalar field has a local maximum.

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