Viable Singularity-Free $f(R)$ Gravity without a Cosmological Constant

Vinícius Miranda, Sergio E. Jorás, and Ioav Waga
Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, Rio de Janeiro, RJ 21941-972, Brazil

Miguel Quartin
INAF/Osservatorio Astronomico di Roma, Via Frascati 33, 00040 Monteporzio Catone, Roma, Italy,
Dipartimento di Fisica "G. Occhialini," Università di Milano-Bicocca, Piazza della Scienza 3, 20126 Milano, Italy,
and Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany
(Received 25 February 2009; published 1 June 2009)

Several authors have argued that self-consistent $f(R)$ gravity models distinct from the cold dark matter model with a cosmological constant (ΛCDM) are almost ruled out. Confronting such claims, we present a particular two-parameter $f(R)$ model that (a) is cosmologically viable and distinguishable from ΛCDM, (b) is compatible with the existence of relativistic stars, (c) is free of singularities of the Ricci scalar during the cosmological evolution, and (d) allows the addition of high-curvature corrections that could be relevant for inflation.

DOI: 10.1103/PhysRevLett.102.221101 PACS numbers: 04.50.Kd, 98.80.−k

Introduction.—Since the discovery of cosmic acceleration, more than a decade ago, considerable effort has been devoted in cosmology to understand what is the physical mechanism responsible for it. A relic cosmological constant Λ, even though arguably the simplest explanation and in good accordance with observations, faces some theoretical difficulties (mainly due to the cosmic coincidence problem and related fine-tuning [1]) that have motivated an intense search for alternatives. These can be divided into two main conceptual approaches, both involving the introduction of new degrees of freedom (see for instance [2]): either one modifies the left-hand side of Einstein's equations (modified gravity) or one adds a new term to the energy-momentum tensor, arguably associated with a new fundamental field not directly related to gravity.

Special attention to the former approach has been given in the past five years. In particular, $f(R)$ gravity theory, due to its simplicity, received the main focus (for a recent review, see [3] and references therein). This approach amounts to writing the action as

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}f(R) + \mathcal{L}_{\rm mat}\right], \tag{1}$$

where $f(R) = R + \Delta(R)$, R is the Ricci scalar, and $\Delta(R)$ is an arbitrary function. General relativity (GR) without a cosmological constant is obtained in the special case in which $\Delta(R)$ is identically zero. Although a great deal of effort has been employed to develop this approach, it appeared to be a difficult challenge to build a new Lagrangian that does not spoil the successes of GR—one that passes solar system tests, describes the early Universe, allows a matter-dominated phase followed by an accelerating attractor [4]—and, at the same time, does not suffer from curvature singularities [5]. The presence of singularities may have devastating consequences and could forbid, for instance, the formation of relativistic stellar objects such as neutron stars [6].

Singularity-free $f(R)$ model.—Several popular $f(R)$ models investigated in the literature are generalized by the following expression:

$$f(R) = R - R_S\beta\left\{1 - \left[1 + \left(\frac{R}{R_*}\right)^n\right]^{-1/\beta}\right\}. \tag{2}$$

For instance, choosing $\beta = -1$ we obtain the models presented in [7]; for $\beta = 1$ we recover the model proposed in [8]; for $n = 2$ we get the $f(R)$ function discussed in [9]. In this Letter we consider the special case in which $n = 1$ and we take the limit $\beta \to \infty$. In this limit (2) can be recast as (rewriting R_S as αR_*)

$$f(R) = R - \alpha R_* \ln\left(1 + \frac{R}{R_*}\right), \tag{3}$$

where α and R_* are free positive parameters. Notice that the above function satisfies the stability conditions [10] (a) $f_{RR} := d^2f/dR^2 > 0$ (no tachyons [11]), (b) $f_R := df/dR > 0$ (no ghosts) for $\alpha < (\tilde{R}/R_* + 1)$, where $\tilde{R}$ is the value of the Ricci scalar at the final accelerated fixed point, and (c) $\lim_{R\to\infty}\Delta/R = 0$ and $\lim_{R\to\infty}\Delta_R = 0$ (GR is recovered at early times). Above and henceforth, $\Delta_R := d\Delta/dR$.

Starting from the action (1), one obtains the equation of motion for $f(R)$:

$$f_R R_{\mu\nu} - \nabla_\mu\nabla_\nu f_R + \left(\Box f_R - \frac{1}{2}f\right)g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{4}$$

the trace of which is given by

$$\Box f_R = \frac{8\pi G}{3}T + \frac{1}{3}(2f - f_R R), \tag{5}$$

where T is the trace of the energy-momentum tensor. We now introduce the scalar degree of freedom (d.o.f.) $\chi := f_R$ and write the equations based on the mapping from $f(R)$ gravity (with positive first and second derivatives) onto Brans-Dicke scalar-tensor theory with parameter $\omega = 0$. The resulting field equation is

$$\Box\chi = \frac{dV}{d\chi} - \mathcal{F}, \tag{6}$$

with the force term given by $\mathcal{F} := -(8\pi G/3)T$ and

$$\frac{dV(R(\chi))}{d\chi} := \frac{1}{3}(2f - f_R R). \tag{7}$$

When applying the model (3) to a spatially homogeneous and isotropic universe, the scalar d.o.f. becomes

$$\chi[R(t)] = 1 - \frac{\alpha R_*}{R(t) + R_*}, \tag{8}$$

and the d'Alembertian in (6) is effectively just a time-derivative: $\Box \equiv -\partial^2/\partial t^2 - 3H\partial/\partial t$; our choice for the metric signature is $(-,+,+,+)$. It is straightforward to see that $\chi \to 1^-$ as $R \to \infty$, which points out the same singularity [5] featured in previous models [8,9,12]. Inverting the relation (8) and integrating (7) we find that (up to a constant)

$$\frac{3V(\chi)}{R_*} = -\alpha(2\chi - 3)\ln\left(\frac{\alpha}{1-\chi}\right) + (\chi - 1)\left(\frac{\chi - 3}{2} - \alpha\right). \tag{9}$$

Note that since (8) defines a one-to-one relation between χ and R, the potential $V(\chi)$ is well defined and not multivalued, contrary to the models in [7–9]. Figure 1 depicts the potential for $\alpha = 2$, as well as typical potentials derived from models [8,9]. Taking the limit $\chi \to 1^-$, we find that

$$V(\chi \to 1^-) \approx \frac{\alpha R_*}{3}\ln\left(\frac{\alpha}{1-\chi}\right) \to +\infty, \tag{10}$$

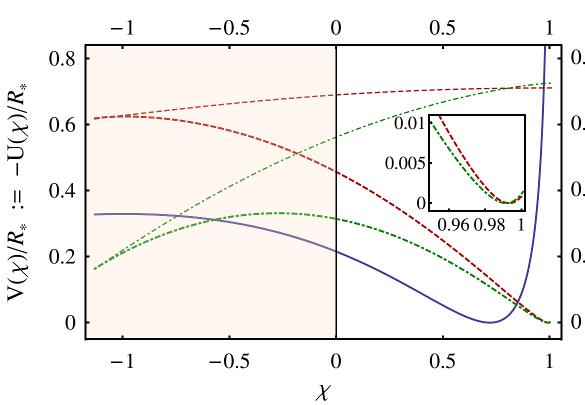

FIG. 1 (color online). $V(\chi)/R_* := -U(\chi)/R_*$ for different models: ours, with $\alpha = 2$ (blue solid line), Starobinski's for $\{n = 2, \lambda = 2\}$ [see [9]] (red dashed line), and Hu and Sawicki's for $\{n = 2, m^2 = 1, c_1/c_2 = 2\}$ [see [8]] (green dot-dashed line). The physically interesting region is $0 < \chi < 1$. For the multi-valued potentials only the lower lines are physical.

which shows the presence of an infinite barrier at $\chi = 1$ that prevents the singularity discussed in [5] to be reached.

We can understand this result in a more intuitive way by making use of the well-known duality between $f(R)$ and scalar-tensor theories: a conformal transformation of the metric can cast the Lagrangian from the Jordan into the Einstein frame, converting the scalar d.o.f. χ into a canonical scalar field $\tilde{\chi} := -\sqrt{3/16\pi G}\ln\chi$ [13]. The field equation for $\tilde{\chi}$ has the same structure of (6), but with the following potential:

$$V_E(R(\tilde{\chi})) = \frac{1}{16\pi G}\frac{R\Delta_R - \Delta}{(1 + \Delta_R)^2}. \tag{11}$$

All the discussion above, regarding the presence of an infinite barrier, applies to V_E as well. Note that since $1 + \Delta_R > 0$ [stability condition (b)], the numerator of Eq. (11) is the only factor that can make the potential diverge as $R \to \infty$. In [14] the singularity was avoided by introducing an extra high-curvature term $\alpha R^{n(>1)}$ in the model investigated in [9]. It is easy to see why that kind of correction works: in that case, the numerator in (11) is itself proportional to R^n. Nevertheless, such term cannot be used, at the same time, both to avoid the singularities and to generate inflation [15]. This is not the case of the model investigated in this Letter, since it is not necessary to include such terms to avoid the two singularity problems, as we have shown above (for the case discussed in [5]) and will show below for the case discussed in [6].

Notice that two different singularity-free classes of $f(R)$ are possible: we can pick a function Δ such that either $\lim_{R\to\infty} R\Delta_R = \infty$ or $\lim_{R\to\infty}\Delta = -\infty$ holds. In the former case, Δ can even become constant as $R \to \infty$—which actually happens in the models previously mentioned [8,9]—but it should do so slowly, thus keeping the divergence of $R\Delta_R$, which does not happen on those models. The model (3) belongs to the latter case. Another interesting example of this class is

$$f(R) = R - \alpha R_*\left(1 + \frac{R}{R_*}\right)^n, \tag{12}$$

with $\alpha > 0$, $R_* > 0$, and $n \in (0, 1)$. Although preliminary tests indicate that this model is cosmologically viable, it carries an explicit positive cosmological constant, in direct contrast to (3).

We further remark that the potential (11) derived from (3) generates a Yukawa-like force which is fully compatible with the chameleon mechanism [13,16]. In other words, the mass of the χ field is large (small) when the background matter density is large (small). This mass dependence on the local environment explains how this extra (or fifth) force can have cosmological implications while at the same time evading detection by local gravity experiments.

Relativistic stars.—The authors of [6] argue that the very existence of relativistic stars poses a strong constraint on $f(R)$ gravity theories. For the models studied in that paper, it was not possible to evolve the metric from inside a star up to large spatial scales and match the de Sitter solution asymptotically. We show below that this divergence is circumvented by model (3) and, therefore, does not represent a general feature of $f(R)$ models. For the sake of clarity, we follow the classical-mechanics analogy used in [6] and the necessary definitions. We consider a static and spherically symmetric metric and write the d'Alembertian in (6) as $\Box \equiv \partial^2/\partial r^2 + (2/r)\partial/\partial r$ in spherical coordinates; we are assuming a Minkowski background for a moment. In this case, Eq. (6) can be seen as the equation of motion of a classical particle of unit mass (albeit one whose "time" coordinate is our spatial coordinate r) submitted to both an external and frictional forces. Therefore,

$$\frac{d^2\chi}{dr^2} + \frac{2}{r}\frac{d\chi}{dr} = \tilde{\mathcal{F}} + \mathcal{F}_U, \qquad (13)$$

where $\tilde{\mathcal{F}} := -\mathcal{F}$ and $\mathcal{F}_U := -dU/d\chi$ are, respectively, the "force" due to the trace of the energy-momentum tensor (nonvanishing inside the star) and the "force" due to the potential $U(\chi) = -V(\chi)$; see Eq. (9) and Fig. 1. Again, the change in sign is just a consequence of the fact that now it is the spatial (instead of time) dependence of χ which is the most relevant.

For the models analyzed in [6], there was no solution which would describe a particle going uphill pulled by the force $\tilde{\mathcal{F}}$ (while still inside the star) and stop at the top of the potential at $r \to \infty$, which would correspond to the de Sitter metric. The particle would either return and reach the singularity at $\chi = 1$ (where $R \to \infty$) or overshoot the potential towards $\chi = 0$ (which would also lead to a singularity, for instance, in the Kretschmann scalar $K := R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}$). Fairly enough, $U(\chi)$ diverges at $\chi = 1$, as in all other models [8,9]. As we will show below, the advantage here is a well-behaved solution fully compatible with relativistic stars embedded in a de Sitter universe.

Let us now determine the full evolution of the χ field. As previously mentioned, we start from a static and spherically symmetric metric

$$ds^2 = -N(r)dt^2 + \frac{1}{B(r)}dr^2 + r^2 d\Omega^2 \qquad (14)$$

and assume a constant energy-density star whose energy-momentum tensor is given by $T^\nu_\mu = \mathrm{diag}(-\rho_0, p(r), p(r), p(r))$. The initial conditions at $r_i = 10^{-8}R_*^{-1/2}$, i.e., close to the center of the star, are given by $N(r_i) = 1 + N_2 r_i^2$, $B(r_i) = 1 + B_2 r_i^2$, $p(r_i) = p_c + p_2 r_i^2/2$, and $\chi(r_i) = \chi_c(1 + C_2 r_i^2/2)$. The coefficients N_2, B_2, p_2, and C_2 can be written in terms of $\rho_0 = 2 \times 10^8 \Lambda_{\mathrm{eff}}$ and of the central values $p_c = 0.3\rho_0$, $R_c = 10^{-8}\rho_0$, $V(\chi_c)$, and $dV/d\chi(\chi_c)$. The effective value of the cosmological constant is given by $\Lambda_{\mathrm{eff}} = R_1/4$, where R_1 is the value of the Ricci scalar when $dV/d\chi = 0$. We refer the reader to the original paper [6] for the full set of equations. Energy conservation provides an important relation between $p(r)$ and $N(r)$ inside the star. We evolve the system $\{p, B, \chi, d\chi/dr\}$ from r_i up to the radius $\mathcal{R}$ of the star [defined by $p(\mathcal{R}) = 0$] where we require continuity of the variables. From then on we evolve the system $\{N, B, \chi, d\chi/dr\}$ until $r = R_*^{-1/2}$ (cosmological scales).

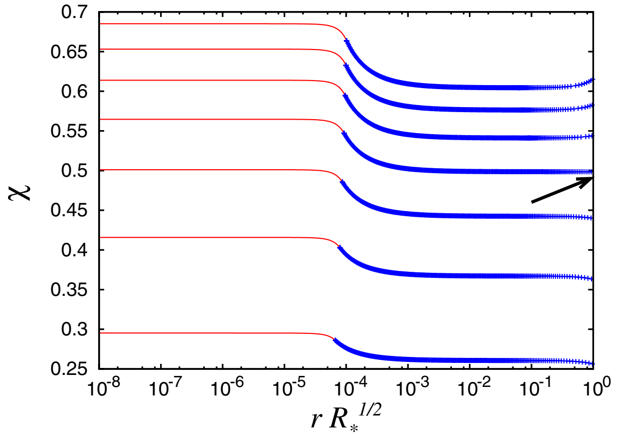

FIG. 2 (color online). The field χ with $\alpha = 1.2$, $p_c = 0.3\rho_0$, and R_c varying from $1 \times 10^{-8}\rho_0$ to $4 \times 10^{-8}\rho_0$. The arrow points out the solution that stops at the maximum of the potential at $r \to \infty$. The thin (red) lines indicate the region inside the star.

We show in Fig. 2 the behavior of the field χ for different values of initial conditions. Note that some trajectories do not get past the top of the potential and return towards the singularity at $\chi = 1$ (top three curves) while others (three lowest ones) overshoot and go towards $\chi = 0$ and one (indicated by an arrow) stops right at the maximum. We recall that this solution was obtained without any high-curvature correction. It is obviously an issue of fine-tuning the initial conditions to stop exactly there. Another remarkable feature of this model is the absence of singularity in K as χ decreases below the peak of its potential.

Promising model.—A viable cosmological model must start with a radiation-dominated universe and have a saddle point matter-dominated phase followed by an accelerated epoch as a final attractor. We can formally state such criteria if we use the parameters $m := Rf_{,RR}/f_R$ and $\mathfrak{r} := -Rf_R/f$. We refer the reader to the original paper [4] for a full discussion on this subject. An early matter-dominated epoch of the Universe can be achieved if $m(\mathfrak{r} \approx -1) \approx 0^+$ and $dm/d\mathfrak{r}(\mathfrak{r} \approx -1) > -1$. Furthermore, a necessary condition for a given model to reach a late-time accelerated phase is $0 < m(\mathfrak{r} \approx -2) \leq 1$. The model (3) satisfies both constraints for $\alpha > 1$ regardless of R_*.

Using (4), we obtain the modified Einstein equations below for a homogeneous Universe filled with matter energy density ρ_m (baryons and cold dark matter) and radiation energy density ρ_r:

$$3H^2 = 8\pi G(\rho_m + \rho_r) + (f_R R - f)/2 - 3H\dot{f}_R + 3H^2(1 - f_R) \qquad (15)$$

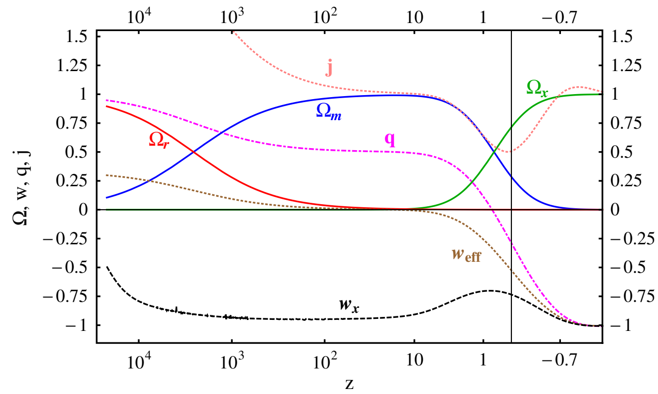

FIG. 3 (color online). Cosmological evolution of the densities Ω_m, Ω_r, Ω_c (solid lines), the deceleration factor q (dot-dashed line), the jerk j (dotted line), the equation-of-state parameters w_x and $w_{\rm eff}$ (dashed line and dotted line, respectively), for $\alpha = 2$.

$$-2\dot{H} = 8\pi G(\rho_m + 4\rho_r/3) + \ddot{f}_R - H\dot{f}_R - 2\dot{H}(1 - f_R), \tag{16}$$

where a dot corresponds to derivative with respect to t, $H \equiv \dot{a}/a$, and $a(t)$ is the scale factor. From the equations above, we can define ρ_x, p_x, and $w_x := p_x/\rho_x$, respectively, the energy density, pressure, and the equation-of-state parameter of the so-called "curvature fluid":

$$8\pi G\rho_x := (f_R R - f)/2 - 3H\dot{f}_R + 3H^2(1 - f_R) \tag{17}$$

$$8\pi G p_x := \ddot{f}_R + 2H\dot{f}_R - (2\dot{H} + 3H^2)(1 - f_R) + (f - f_R R)/2. \tag{18}$$

These definitions are such as to guarantee that the curvature fluid is conserved and only minimally coupled to matter and radiation [17]. We also define the relative densities Ω_i (where i stands for either radiation, matter, or curvature) $\Omega_i := 8\pi G\rho_i/3H^2$.

In Fig. 3 we plot the behavior of Ω_m, Ω_r, Ω_x, the deceleration parameter $q := -\ddot{a}a/\dot{a}^2$, the jerk $j := \dddot{a}a/\dot{a}^2$, the equation-of-state parameters for the curvature fluid w_x and for the effective fluid $w_{\rm eff} := p_{\rm tot}/\rho_{\rm tot} \equiv (p_r + p_x)/(\rho_m + \rho_r + \rho_x)$, all of which can be written in terms of known variables R, H^2, and ρ_i. In Fig. 3 we can clearly distinguish the radiation-dominated era when $q \simeq 1$ (and $j \simeq 3$, not shown), followed by a transient domination by matter ($q \simeq 1/2$ and $j \simeq 1$), the current accelerated expansion ($q < 0$), and the final de Sitter attractor ($q = -j = -1$). We find similar results for different initial conditions and parameters, indicating what seems to be an absence of fine-tuning. We remark that the w_x curve in Fig. 3 is noisy in the early Universe since at that time ρ_x is too small and the numerical calculation of w_x becomes inaccurate.

We point out that there is some residual arbitrariness in defining ρ_x and p_x even if one is only interested in conserved and minimally coupled fluids. The one we follow, together with the definition of Ω_i, is convenient for comparison with GR-based interpretations of observations. As another consequence of Eqs. (17) and (18), w_x neither crosses -1 nor diverges at low redshift in contrast with [18], where slightly different definitions are adopted. Note, however, that observable quantities like H and ρ_m are well defined and in fact, using either definition, have the same cosmological evolution.

Conclusions.—We have shown that some recent results in the literature regarding divergences in $f(R)$ theories are not as general as previously thought. In fact, even a compact, two-parameter Lagrangian like the one in (3) can evade the aforementioned singularities. Observational constraints on this model are under investigation and the results will be published elsewhere.

We thank Luca Amendola, Philippe Brax, and Ribamar R. R. Reis for helpful discussions in the course of this work. V. M. and I. W. acknowledge financial support from the Brazilian Research Agency CNPq. M. Q. thanks OAR, U. Milano-Bicocca, and ITP U. Heidelberg for support.

[1] M. Quartin, M. O. Calvão, S. E. Jorás, R. R. R. Reis, and I. Waga, J. Cosmol. Astropart. Phys. 05 (2008) 007.
[2] J.-P. Uzan, Gen. Relativ. Gravit. **39**, 307 (2007).
[3] T. P. Sotiriou and V. Faraoni, arXiv:0805.1726v2 [Rev. Mod. Phys. (to be published)].
[4] L. Amendola, R. Gannouji, D. Polarski, and S. Tsujikawa, Phys. Rev. D **75**, 083504 (2007).
[5] A. V. Frolov, Phys. Rev. Lett. **101**, 061103 (2008).
[6] T. Kobayashi and K. I. Maeda, Phys. Rev. D **78**, 064019 (2008).
[7] S. Capozziello, V. F. Cardone, S. Carloni, and A. Troisi, Int. J. Mod. Phys. D **12**, 1969 (2003); S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).
[8] W. Hu and I. Sawicki, Phys. Rev. D **76**, 064004 (2007).
[9] A. A. Starobinsky, JETP Lett. **86**, 157 (2007).
[10] L. Pogosian and A. Silvestri, Phys. Rev. D **77**, 023503 (2008).
[11] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573**, 1 (2003); V. Faraoni, Phys. Rev. D **74**, 104017 (2006).
[12] S. A. Appleby and R. A. Battye, J. Cosmol. Astropart. Phys. 05 (2008) 019; Phys. Lett. B **654**, 7 (2007).
[13] P. Brax, C. van de Bruck, A. C. Davis, and D. J. Shaw, Phys. Rev. D **78**, 104021 (2008).
[14] A. Dev, D. Jain, S. Jhingan, S. Nojiri, M. Sami, and I. Thongkool, Phys. Rev. D **78**, 083515 (2008).
[15] T. Kobayashi and K. I. Maeda, Phys. Rev. D **79**, 024009 (2009).
[16] J. Khoury and A. Weltman, Phys. Rev. Lett. **93**, 171104 (2004); Phys. Rev. D **69**, 044026 (2004).
[17] F. Perrotta and C. Baccigalupi, Phys. Rev. D **65**, 123505 (2002); D. F. Torres, Phys. Rev. D **66**, 043522 (2002).
[18] L. Amendola and S. Tsujikawa, Phys. Lett. B **660**, 125 (2008).