Properties of the Photonic Hall Effect in Cold Atomic Clouds

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On the basis of exact numerical simulations and analytical calculations, we describe qualitatively and quantitatively the interference processes at the origin of the photonic Hall effect for resonant Rayleigh (point-dipole) scatterers in a magnetic field. For resonant incoming light, the induced giant magneto-optical effects result, even for magnetic field strength as low as a few mT, in relative Hall currents in the percent range. This suggests that the observation of the photonic Hall effect in cold atomic vapors is within experimental reach.

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Light propagation in homogeneous media in the presence of a static magnetic field is a rich and vivid field of research where the symmetries dictated by nature lead to subtle magneto-optical phenomena [1]. About ten years ago, the question of magneto-optics in strongly scattering media was addressed, and several effects bearing close analogies with electronic transport were theoretically predicted and observed [2]. One striking example is the photonic Hall effect (PHE) where light propagating in a scattering medium subject to a transverse magnetic field can be deflected in the direction perpendicular to both the incident beam and the magnetic field [3–5]. Similar magnetotransverse effects have also been observed with (2D) polaritons [6] and phonons [7]. Cold atomic gases provide an appealing testing ground for the PHE in the multiple scattering regime. Indeed, they constitute a perfect monodisperse sample of highly resonant point scatterers, with almost no spurious phase-breaking mechanisms. In addition, few 10^{-4} Teslas (T) are enough to induce strong magneto-optical effects like the Faraday rotation [8] in sharp contrast with classical materials. If the impact of a magnetic field on coherent backscattering (CBS) has already been studied [9,10], the question of the observation of the PHE in atomic vapors is still open. In this Letter, we present analytical and numerical calculations identifying the physical origin of the PHE for point-dipole scatterers. Our results show that the effect should be observable in cold atomic gases.

A quantitative study of the PHE needs to address the question of directional asymmetries displayed by the configuration-averaged radiation pattern of an assembly of atoms located at random positions and illuminated by an incident monochromatic plane wave (wave vector $\mathbf{k}$, polarization vector $\boldsymbol{\epsilon} \perp \mathbf{k}$, angular frequency $\omega = ck = 2\pi c/\lambda$) while being subjected to an external static magnetic field with strength B pointing in the direction $\hat{\mathbf{B}}$. We consider here two-level atoms having a ground state with angular momentum $J = 0$ connected by a narrow optical dipole transition to an excited state with angular momentum $J_e = 1$. The energy separation between the atomic states is $\hbar\omega_0 = hc/\lambda_0$, and the natural energy width of the excited state is $\hbar\Gamma \ll \hbar\omega_0$. This is one of the best possible natural realizations of resonant point scatterers [11], and it corresponds, for example, to the case of ^{88}Sr atoms ($\lambda_0 = 461$ nm, $\Gamma/2\pi = 32$ MHz, Landé factor of the excited state $g_e = 1$). When $B = 0$, the incident light is quasiresonant ($\lambda \approx \lambda_0$) with this optical dipole transition, and we will denote by $\delta = (\omega - \omega_0)$ the light detuning with respect to the atomic line ($\delta \ll \omega_0$). We assume the incident light intensity to be low enough to neglect all nonlinear effects. When the magnetic field is applied, the internal degeneracy is lifted (Zeeman effect) and the excited level is split into 3 components separated by μB, where $\mu/2\pi = 14$ GHz/T is the Zeeman shift rate. As soon as the Zeeman shift becomes comparable to the resonance width, i.e., $\phi_B = 2\mu B/\Gamma \sim 1$, the scattering properties of each atom are strongly modified (at $B \sim 1.1$ mT in the case of ^{88}Sr).

The source of the field radiated by the atom is the oscillating electric dipole moment $\mathbf{d}\exp(-i\omega t)$ induced by the incident electric field $\mathbf{E}\exp(-i\omega t)$. The radiated spectrum is here elastic as there is no Zeeman effect in the ground state. The situation would be more involved for atoms with a degenerate ground state where frequency changes lead to an inelastic spectrum. In our situation, $\mathbf{d} = \epsilon_0 \underline{\boldsymbol{\alpha}}(\mathbf{B})\mathbf{E}$ and the radiation properties are fully characterized by the polarizability tensor $\underline{\boldsymbol{\alpha}}(\mathbf{B}) = \alpha_0 \mathcal{T}(\mathbf{B})$ given by

$$\mathcal{T}(\mathbf{B}) = \zeta(B)\mathbb{1} + \eta(B)\mathbb{1} \times \hat{\mathbf{B}} + \xi(B)\hat{\mathbf{B}}\,\hat{\mathbf{B}}, \qquad (1)$$

where $\alpha_0 = \frac{6\pi}{k^3}\frac{\Gamma/2}{\delta + i\Gamma/2}$ is the complex atomic polarizability at $B = 0$. The dyadic tensor $\mathcal{T}(\mathbf{B})$ embodies the usual magneto-optical effects induced on the photon polarization degrees of freedom. The ζ term is responsible for the normal extinction of the forward beam (Lambert-Beer law). The η term describes the magnetically induced rotation of the atomic dipole moment (Hanle effect) [9] and

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induces Faraday rotation and dichroism effects in the forward beam when $\mathbf{k} \parallel \mathbf{B}$ [1]. Finally, the ξ term is responsible for the Cotton-Mouton effect also observed in the forward beam when $\mathbf{k} \perp \mathbf{B}$ [1]. These coefficients read

$$\phi = \frac{\phi_B}{1 - 2i\delta/\Gamma}, \qquad \zeta = \frac{1}{1+\phi^2}, \qquad \eta = \frac{\phi}{1+\phi^2}, \qquad \xi = \frac{\phi^2}{1+\phi^2} \tag{2}$$

and are real on resonance ($\delta = 0$). We get the single-atom differential cross section from the polarizability:

$$I(\mathbf{k}\boldsymbol{\epsilon} \rightarrow \mathbf{k}'\boldsymbol{\epsilon}') = \frac{k^4}{16\pi^2}|\bar{\boldsymbol{\epsilon}}'\underline{\alpha}(\mathbf{B})\boldsymbol{\epsilon}|^2 = \frac{3\sigma_0}{8\pi}|\bar{\boldsymbol{\epsilon}}'\mathcal{T}\boldsymbol{\epsilon}|^2, \tag{3}$$

where $\sigma_0 = |\alpha_0|^2 k^4/6\pi$ is the total scattering cross section at zero magnetic field. As immediately seen, $I(\mathbf{k}\boldsymbol{\epsilon} \rightarrow \mathbf{k}'\boldsymbol{\epsilon}')$ depends only on the incoming and outgoing polarizations and is thus completely insensitive to reversing the direction of observation $\mathbf{k}' \rightarrow -\mathbf{k}'$ [12,13]. As a consequence, there is no directional asymmetry in the single scattering signal radiated by an assembly of such atoms, and the PHE, if any, must come from a multiple scattering effect.

Before considering the general case, we first analyze the radiation properties of two isolated atoms separated by the relative vector $\mathbf{r}$. This is the simplest possible situation where multiple scattering plays a role. In the Hall geometry (see Fig. 1), the differential cross sections $I_\pm$ are measured along the up and down directions $\pm\hat{\mathbf{y}}$ perpendicular both to the incident light direction $\mathbf{k} = k\hat{\mathbf{x}}$ and to the magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The total radiation field is the sum of the fields radiated by the atomic dipoles induced by the incoming and scattered fields at their respective positions. The exact solution involves the inversion of a linear system of 2 coupled vectorial equations where the polarizability tensor plays a key role. We compute exactly the differential cross sections $I_\pm$ for a fixed relative distance r between the atoms and average them over all possible relative orientations of the atoms to get $\langle I_\pm \rangle$. We then extract the Hall current $\Delta I = \langle I_+ \rangle - \langle I_- \rangle$, the mean intensity $I = (\langle I_+ \rangle + \langle I_- \rangle)/2$, and the relative Hall current $\beta = \Delta I/I$. There are 4 possible linear polarization channels $\boldsymbol{\epsilon} \rightarrow \boldsymbol{\epsilon}'$ for the data analysis. However, as we have numerically checked, the Hall current *must be the same* in the linear channels $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ and $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{x}}$ (the channels are related by time-reversal symmetry) and *must vanish* in the $\hat{\mathbf{z}} \parallel \hat{\mathbf{z}}$ channel (the polarizations are along the magnetic field and insensitive to the Hanle effect). We are thus left with the two lin $\perp$ lin channels $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ and $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$. Figure 2 summarizes our numerical results. The Hall current vanishes as $kr \rightarrow 0$. Indeed, for very small distances, the two radiating dipoles are always in phase and add up constructively. The two atoms behave like a single scatterer with a dipole moment twice larger and cannot display directional asymmetries. For low ϕ values, the relative Hall current β decreases in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ channel, whereas, in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ channel, it is comparable to the one for high ϕ values. Indeed, in the latter channel, the background intensity gets more and more contaminated (and even dominated at large distances) by the single scattering signal induced by the Hanle effect when the magnetic field is increased. In the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ channel, single scattering is always filtered out.

To gain insight about the physical processes at work, we consider the case of a "dilute" medium $kr \gg 1$, and we expand the field radiated by the two atoms in powers of $(kr)^{-1}$. Skipping tedious details, the scattered amplitude is obtained at leading order from the sum of the two diagrams with respective amplitude u and v shown in Fig. 3(a). Each differential cross section $I_\pm$ contains interference terms (i.e., $u\bar{v} + v\bar{u}$) and background terms (i.e., $|u|^2 + |v|^2$). The latter cancel out in ΔI as they do not depend on the scattering direction, and the Hall current is solely given by a difference of interference terms, which are precisely the

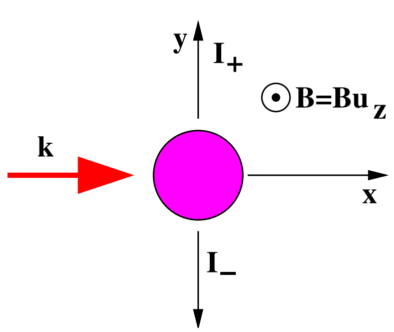

FIG. 1 (color online). The photonic Hall geometry. A plane wave $\mathbf{k} = k\hat{\mathbf{x}}$ is scattered by a cloud of atoms subjected to a static magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The Hall current is measured either in the linear polarization channel $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ or in the $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ one. It is defined as $\Delta I = \langle I_+ \rangle - \langle I_- \rangle$, with $\langle I_\pm \rangle$ the configuration-averaged differential cross section along $\pm\hat{\mathbf{y}}$.

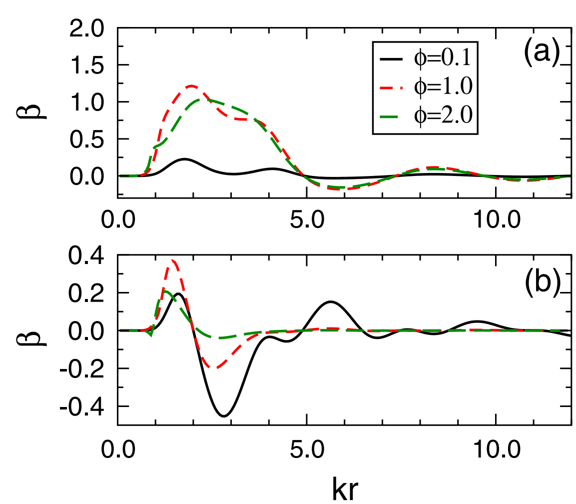

FIG. 2 (color online). Two-atom case. Relative Hall current β observed at resonance $\delta = 0$ in the linear polarization channels (a) $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{z}}$ and (b) $\hat{\mathbf{y}} \rightarrow \hat{\mathbf{x}}$ as a function of the relative distance kr for different values of $\phi_B = 2\mu B/\Gamma$.

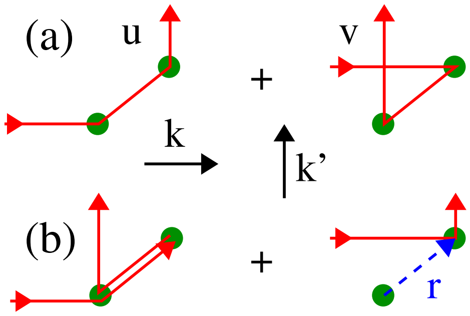

FIG. 3 (color online). Two-atom case. For large distances, the PHE in the $\hat{\mathbf{y}} \to \hat{\mathbf{z}}$ channel results from the interference between the two diagrams (a). In the $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$ channel, an additional contribution comes from the interference between the two diagrams accounting for recurrent scattering (b) (and also between the ones obtained by exchanging the two atoms).

crossed terms at the heart of the CBS effect [14]. More precisely, the Hall current reads

$$\Delta I = \langle \delta I(\mathbf{r}, \mathbf{B})\{\cos[(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}] - \cos[(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}]\}\rangle, \tag{4}$$

where $\delta I(\mathbf{r}, \mathbf{B}) \propto |\bar{\boldsymbol{\epsilon}}' \mathcal{T}(\phi) \boldsymbol{\Delta}_{\mathbf{r}} \mathcal{T}(\phi) \boldsymbol{\epsilon}|^2$, $\langle \ldots \rangle$ denotes the average over the relative orientation of the two atoms, and $\boldsymbol{\Delta}_{\mathbf{r}}$ is the projector onto the plane perpendicular to $\mathbf{r}$. Equation (4) can be easily recast as

$$\Delta I = \langle [\delta I(\mathbf{r}, \mathbf{B}) - \delta I(\mathbf{r}, -\mathbf{B})] \cos[(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}]\rangle. \tag{5}$$

The Hall current can then be understood as a difference between the two configuration-averaged interference effects generated in the *same* direction $\mathbf{k}'$ for opposite directions of the magnetic field. The imbalance results from the interplay between the dipole rotation induced by the Hanle effect and the transverse projector $\boldsymbol{\Delta}_{\mathbf{r}}$. As the cosine term oscillates at the optical wavelength scale, a stationary phase approximation shows that the main contribution in Eq. (5) comes from configurations where the two atoms are aligned along the $\mathbf{k} + \mathbf{k}'$ direction.

Performing the angular average, one gets in the $\hat{\mathbf{y}} \to \hat{\mathbf{z}}$ channel

$$\Delta I \approx \frac{81}{8k^2} \frac{\phi_B}{|(1 - 2i\delta/\Gamma)^2 + \phi_B^2|^2} \frac{\cos(\sqrt{2}kr)}{(kr)^4}. \tag{6}$$

The $\sqrt{2}kr$ term corresponds to the value of $(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}$ when these two vectors are parallel.

In the $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$ channel, at the same order, there is an additional contribution due to diagrams accounting for the interference between recurrent and single scattering processes [see Fig. 3(b)]. The final expression is quite tedious but simplifies at resonance $\delta = 0$:

$$\Delta I \approx \frac{81\sqrt{2}}{8k^2} \frac{\phi_B}{(1 + \phi_B^2)^3} \frac{\sin(\sqrt{2}kr)}{(kr)^3} [1 + \cos(2kr)]. \tag{7}$$

As shown in Fig. 4, for $\delta = 0$ and $\phi_B = 1$, the agreement between the numerical computations and these asymptotic results proves excellent as soon as $kr \gtrsim 10$. Quite remarkably, recurrent scattering is essential to reproduce the additional oscillations observed in the $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$ channel and due to the $\cos 2kr$ term in Eq. (7). We have numerically checked that the agreement is still excellent in both channels when $\delta \neq 0$. This gives clear evidence that the physical mechanism at the heart of the PHE for atomic scatterers is the interference between the scattering processes depicted by the diagrams in Fig. 3. In addition, the fact that the ΔI is proportional to ϕ_B (i.e., to the atomic dipole rotation) in the small ϕ_B limit actually implies that only one of the two scatterers needs to be magnetoactive to generate a Hall current. Finally, contrary to the case studied in Ref. [3], both our analytical and numerical results show that the PHE is still present when the antisymmetric part of the self-energy $[\Sigma \propto \underline{\boldsymbol{\alpha}}(\mathbf{B})]$ has a vanishing real part (i.e., at resonance $\delta = 0$), the exact dependence on δ being a smooth bell-shaped curve.

For isotropic propagation in a real medium with mean free path $\ell \gg \lambda$, Eqs. (6) and (7) should give a good quantitative estimate of the double scattering contribution to the Hall current once appropriately averaged over all possible distances of the two scatterers. As ΔI exhibits decreasing oscillations at the wavelength scale and since $\lambda \ll \ell$, the main contribution to the Hall current comes from scatterers separated by $r \sim \lambda$. The probability of finding two such scatterers scales with the optical density $\rho = n\lambda^3$, n being the scatterer density. The actual relative Hall current β should thus be smaller than the average background by a factor $(k\ell)^{-1}$ [15], the exact result depending on the exact geometry of the medium (shape and

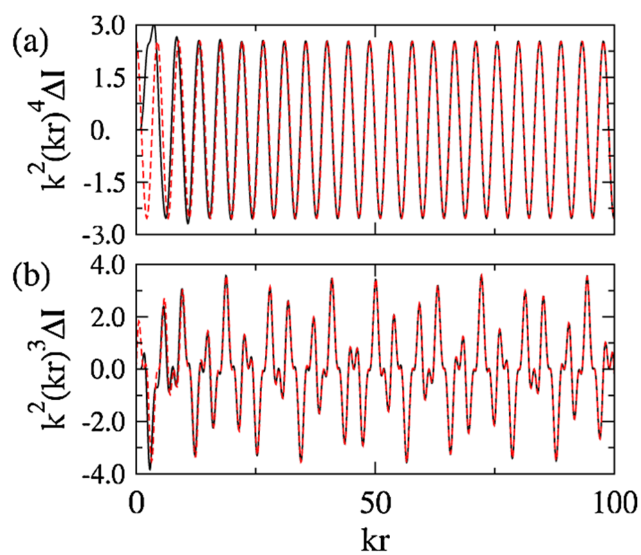

FIG. 4 (color online). Normalized Hall current in the two-atom case. Comparison between the numerical calculations (plain curves) and the analytical predictions Eqs. (6) and (7) (dashed curves) as a function of kr for $\delta = 0$ and $\phi_B = 1$. (a) $\hat{\mathbf{y}} \to \hat{\mathbf{z}}$ channel. (b) $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$ channel. The excellent agreement emphasizes that the underlying physics of the PHE is fully captured by the diagrams depicted in Fig. 3.

TABLE I. The left side displays the relative Hall currents β at $\delta = 0$ and $\phi_B = 1$, in the Hall geometry, for 500 atoms uniformly distributed inside a sphere with optical density $\rho \approx 0.65$. The number of disorder configurations is 6×10^5. The expected accuracy of the order of 10^{-3} is confirmed when comparing to the relative currents β (see right side) in the $\mathbf{k} \parallel \mathbf{B}$ configuration, where no PHE should show up.

$\mathbf{k} \perp \mathbf{B}$		$\mathbf{k} \parallel \mathbf{B}$	
$\hat{\mathbf{z}} \to \hat{\mathbf{x}}$	0.0579	$\hat{\mathbf{x}} \to \hat{\mathbf{z}}$	0.002 67
$\hat{\mathbf{y}} \to \hat{\mathbf{z}}$	0.0545	$\hat{\mathbf{x}} \to \hat{\mathbf{x}}$	−0.000 14
$\hat{\mathbf{y}} \to \hat{\mathbf{x}}$	−0.0088	$\hat{\mathbf{y}} \to \hat{\mathbf{z}}$	−0.001 21
$\hat{\mathbf{z}} \to \hat{\mathbf{z}}$	0.0022	$\hat{\mathbf{y}} \to \hat{\mathbf{x}}$	−0.000 04

size). Nevertheless, the PHE in an assembly of resonant point scatterers, even if rather small, should be measurable.

For numerical confirmation, we have considered $N = 500$ atoms uniformly distributed inside a sphere with diameter D at an optical density $\rho \approx 0.65$ and illuminated by a plane wave set on resonance. This leads to $k\ell = 4\pi^2/3\rho \approx 20$ at $B = 0$. Such a value for $k\ell$ is difficult to achieve in a real experiment but has already been obtained [16]. Light propagation inside the sphere occurs in the multiple scattering regime since the optical thickness of the medium $b = D/\ell \approx 3.5$ is larger than unity. The magnetic field value has been set at $\phi_B = 1$, and the total radiated field has been obtained by solving the corresponding system of $3N$ linear equations [17]. The various quantities of interest are averaged over 6×10^5 configurations leading to an accuracy of about 10^{-3}. To stress the existence of the PHE, we have computed β in the geometry $\mathbf{k} \parallel \mathbf{B}$ where no effect should occur. The right side of Table I shows that the corresponding values are indeed at most about a few 10^{-3}. Up to this accuracy, the numerical results in the geometry $\mathbf{k} \perp \mathbf{B}$ are in qualitative agreement with the two-atom case: There is no Hall current in the $\hat{\mathbf{z}} \to \hat{\mathbf{z}}$ channel. It is about the same order of magnitude in the two conjugate channels $\hat{\mathbf{y}} \to \hat{\mathbf{z}}$ and $\hat{\mathbf{z}} \to \hat{\mathbf{x}}$ ($\beta \approx 5.5\%$). Finally, it is larger in these channels than in the $\hat{\mathbf{y}} \to \hat{\mathbf{x}}$ channel ($|\beta| \approx 1\%$). To enforce the validity of our mesoscopic description, we have computed β for different values of $k\ell$ at fixed optical thickness b. Within the statistical errors, the product $\beta k\ell$ is independent of $k\ell$; see Table II. Note that the values found here are in the percent range at least larger by 1 order of magnitude than what is observed with classical scatterers [4–6], although the magnetic field is smaller by 2 or 3 orders of magnitude. This is because the Zeeman effect in highly resonant atomic scatterers induces a "giant" dipole rotation which enhances the PHE.

In summary, on the basis of numerical and analytical calculations, we have qualitatively and quantitatively explained the underlying interference effect at the origin of the photonic Hall effect for resonant point-dipole scatterers. The effect, albeit small, should be observable in cold atomic vapors. Further investigations would consist in developing the diagrammatic analysis to an arbitrary number of scattering events and in accounting for internal degeneracies in the atomic ground state. A possible extension of the work would address the photonic magnetoresistance effect [18], giving a quantitative and comprehensive description of the photonic Hall effect in cold atomic clouds.

TABLE II. Relative Hall current β in the $\hat{\mathbf{z}} \to \hat{\mathbf{x}}$ and $\hat{\mathbf{y}} \to \hat{\mathbf{z}}$ channels at a fixed optical thickness $b = 1.3$. Within the numerical accuracy, the results indicate that β scales like $1/(k\ell)$.

$k\ell$	40.00	63.25	89.44
$\beta \times k\ell$	4.0(1)	3.8(3)	4.0(3)

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