Symmetry Constraints on the Dynamics of Magnetically Confined Plasma

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In respect of their symmetry properties, toroidal magnetically confined plasmas have much in common with the Taylor-Couette flow. A symmetry-based analysis (equivalent bifurction theory) has proved very powerful in the analysis of the latter problem. This Letter discusses the applicability of the method to nuclear fusion experiments such as tokamaks and pinches. The likely behavior of the simplest models of rotationally symmetric tokamaks is described, and found to be potentially consistent with observation.

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Introduction.—The most developed of the modern magnetic fusion concepts is the tokamak; see, e.g., Ref. [1]. After 40 years of research, a huge amount is known about tokamak behavior. However, a complete understanding of some of the most prominent phenomena has not yet been achieved. The disparate time and spatial scales mean the problem will remain very computationally demanding for the foreseeable future, whereas diagnosing the behavior of some aspects of extremely hot plasma is still a very challenging problem for the experimenter.

In these circumstances, it is natural to consider analogous experiments involving liquids. Seemingly, the most intensively studied configuration most analogous to toroidal plasma devices, is the Taylor-Couette (T-C) experiment. This consists of a tall, annular cylinder of fluid confined by vertical (curved) walls which may rotate independently at different, fixed speeds. There are four external parameters (two rotation rates and two radii of cylinders) plus those determined by the properties of the fluid such as its viscosity, see, in particular, Ref. [2] Sect. 6. The experiment has also been described both in the general literature [[3] Sect. 5] and by Feynman [[4] Sect. 41 6].

For low rotation rates and reasonable values of the other parameters, T-C flow consists of a steady motion in the azimuthal direction which varies only in the radial direction. However, as rotation rates increase, this symmetric state becomes unstable to a wide variety of different modes depending on the particular values of the parameters. The analogy with the tokamak should begin to become apparent: in its basic form as an Ohmically heated torus with circular cross-section, the tokamak is also a four-parameter system, with imposed magnetic field and total current providing the driving energy rather than cylinder rotation, and two geometrical parameters of major and minor radius, where the latter is analogous to fluid layer thickness.

The bifurcating modes in the T-C flow may be classified in terms of their symmetries. All possible symmetries may be discovered by studying the group of symmetries of the experiment $G_S(\mathrm{T-C})$. From Ref. [2] Sect. 6, this group is $SO(2) \times (SO(2) \rhd\!\lhd Z(2))$ which may be written $SO(2) \times O(2)$, where $O(2) = SO(2) \rhd\!\lhd Z(2)$. $SO(2)$ is the rotation group, $Z(2)$ is the reflection group and the symbol $\rhd\!\lhd$ denotes the semidirect product. Possible mode patterns are given by the isotropy subgroups of $G_S(\mathrm{T-C})$, which are numerous. Moreover, when the mathematical analysis showed that certain allowed patterns had not so far been described experimentally, newer experiments were performed which successfully exhibited these symmetries.

Symmetries of the tokamak.—The importance of T-C flow for magnetic fusion is that a group with very similar, but not *identical* structure, namely $(SO(2) \times SO(2)) \rhd\!\lhd Z(2) = G_S(\mathrm{Tok})$ is the group of symmetries of the "periodic cylinder" magnetohydrodynamic (MHD) model of the tokamak and magnetic pinch. The periodic cylinder is a circular cylinder with its flat ends identified one with another, designed to approximate a large aspect ratio torus, ie. one with major radius much larger than minor radius. There are two $SO(2)$ subgroups corresponding to rotations in the two angular coordinates θ and ϕ and a reflection symmetry $(\theta, \phi) \to (-\theta, -\phi)$. To understand the latter symmetry, it is helps to remember that the (rate of change of) current in poloidal field coils generally produces the plasma current in the tokamak, so that device operation is fundamentally controlled by two orthogonal vector fields, the currents in, respectively, the toroidal and poloidal field coils. Since the single-fluid MHD equations are invariant under change of sign of magnetic field, reversing the current in both sets of coils leads to the same dynamics.

Analogy with the T-C flow suggests the tokamak will exhibit a wide variety of behavior as parameters are varied. In practice, the baseline H-mode operation for present ITER tokamak experiments [5] is planned on the basis of a central sawtooth mode, the frequent occurrence of edge localized modes (ELMs), the possible occasional presence of other large scale "tearing" modes, and anomalous heat loss caused apparently by many small-scale modes.

$G_S(\mathrm{T-C})$ and $G_S(\mathrm{Tok})$, although slightly different, have at least three subgroups in common, namely $SO(2)$, $Z(2)$ and $O(2)$. Therefore, potential analogues between phenomena in T-C flow and tokamaks are listed in Table I, on the basis of their respective symmetries. The possible

TABLE I. Tokamak and T-C flow analogues.

Taylor-Couette	Tokamak
Steady sheared flow	MHD equilibrium
Rolls (Taylor cells)	Sawtooth oscillation
Rotating wave	Mirnov oscillation
Modulated rotating waves	Complex Mirnov signal

link between the sawtooth oscillation and steady rolls is discussed below. The association has to be tentative not only because the groups differ, but also because the evolution of the tokamak design has been away from circular cross-section, so that real devices deviate significantly from poloidal (θ) rotational symmetry.

The reflection combines with the azimuthal (ϕ) rotational symmetry so that noncircular tokamaks may be treated with certainty as having only $O(2)$ symmetry. Moreover, individual particle motion is *not* invariant under reversal of sign of magnetic field, so a less collisional "kinetic" plasma (e.g., to which a two-fluid MHD model applies) may not have the $Z(2)$ symmetry property.

Equivariant bifurcation theory.—Equivariant bifurcation theory means bifurcation theory analyses performed in the presence of symmetry [6,7]. Bifurcation theory is the mathematical theory of the onset of instability in nonlinear systems, see, for example, Kuznetsov [8]. The key idea is that near onset, system behavior is governed by a small number of coupled ordinary differential equations (ODEs), with nonlinear interactions among the variables represented by low order terms in a multivariable Taylor expansion. It is a natural generalization of linear stability theory which can be regarded as a truncation of the Taylor series at first order.

Frequently the time dependent variables represent mode amplitudes, and an example frequently quoted concerning the effect of symmetry is when the problem is invariant under reflection. For then both $(+a)$ and $(-a)$ must be solutions of the ODEs, which rules out terms such as a^2 in the governing equations, which do not change sign when a does.

Magnetic fusion experiments might be expected to be a fertile ground for bifurcation theory, since typically the performance is optimal close to the onset of instability. Nonetheless, there is the objection that the radius of convergence of the Taylor series may, if plasma viscosity is neglected, be of order S^{-2}, where S is the Lundquist number, and values of S range up to 10^{12}. It is therefore conceivable that the range of validity of the Taylor series approximation is too small to be quantitatively useful. However, the qualitative predictive powers of bifurcation theory are usually good until another instability emerges, which is why the theory emphasizes qualitative (or more formally topological) properties. Moreover, as far as quantitatively interpreting experiment is concerned, it is conceivable that a renormalization approach may be adequate, e.g., using a low order rational polynomial to represent the neglected higher order terms. There is the caution that when the spatial dependence of the unstable mode changes as fast as it grows, e.g., as occurs in the simplest MHD model of $m > 1$ tearing modes [9], even renormalization may not be enough to relate observations quantitatively to mode amplitude. Qualitative behavior should be the same for these modes, however, and in any event, the relation of the simplest theories to experiment is unclear, because they do not include thermal effects which are now believed to be vital for interpreting experiment.

Axisymmetry.—From the earlier discussion, a first analysis of tokamaks using equivariant bifurcation theory need only assume the $SO(2)$ symmetry in the azimuthal direction, applicable whether a particle or fluid model is appropriate. Introduce an explicit spatial dependence, by supposing that the angle about the axis of symmetry is ϕ, then symmetry-breaking solutions $y(t)$ may be written

$$y = a\exp(in\phi) + \bar{a}\exp(-in\phi). \tag{1}$$

Here the overbar denotes complex conjugate, n is (integer) mode number and $a(t)$ is the time dependent complex mode amplitude.

The aim is to produce low order polynomial nonlinear equations which are invariant under rotation. It will be seen that such an evolution equation for a may not contain quadratic terms, ie. any of the terms a^2, $\bar{a}^2$, or $a\bar{a}$. For, translating the angle ϕ by p/n in Eq. (1) shows that if a gives a solution to the problem, ae^{ip} must also be a solution. However, the quadratic terms acquire factors of either $e^{\pm 2ip}$ or $e^{i0} = 1$. Similarly, cubic terms such as a^3 and $\bar{a}^3$ are excluded. Hence the governing equation for a to cubic accuracy is of Landau form

$$\dot{a} = \mu a + \sigma|a|^2 a, \tag{2}$$

where μ and σ are complex constants.

Equation (2) is easier to understand if the representation $a = r\exp(i\xi)$ is introduced where r is the (real) amplitude of complex number a and ξ is its phase. Differentiating

$$\dot{a} = (\dot{r} + ir\dot{\xi})\exp(i\xi) \tag{3}$$

substituting in Eq. (2) multiplied by $\exp(-i\xi)$, writing $\mu = \mu_r + i\mu_i$ and $\sigma = \sigma_r + i\sigma_i$, and equating real and imaginary parts gives

$$\dot{r} = \mu_r r + \sigma_r r^3, \tag{4}$$

$$\dot{\xi} = \mu_i + \sigma_i r^2. \tag{5}$$

Equation (4) shows that the amplitude r will have the sudden switch-on typical of the Hopf bifurcation as μ_r increases through zero, and if $\sigma_r < 0$ will saturate at finite amplitude. Concerning Eq. (5), note that there is no restriction on the size of μ_i, unlike μ_r which must be small near the bifurcation point. Hence, for small r, the solution y now contains the multiplicative term $\exp(i\mu_i t)$; i.e., the solution is generically of traveling-wave type. Such traveling waves are expected in dissipative systems on general symmetry grounds [10], where it is also argued that the rotating waves will become unstable to modulated traveling waves [10]. The limitations of symmetry arguments are, however, evident in that there is no constraint on the sign of σ_i—the mode frequency may increase or decrease with mode amplitude.

This is a convenient point to comment upon the importance of the hitherto neglected $Z(2)$ symmetry. If ϕ is imagined to correspond to a linear combination of θ and ϕ, then Eq. (1) with ϕ replaced by $-\phi$ is also a solution. This implies that $a = \bar{a}$; hence, a is real and y corresponds to the saturated helical waves (representing tearing modes) expected in single-fluid MHD. Confidence in the applicability of Eq. (2) when a is real may be increased when it is realized that it also appears in Ref. [11] for a detailed analysis of the $m = n = 1$ resistive mode, see additionally Ref. [12], Appendix.

Variational constraint.—There is a further constraint which may well be relevant to tokamaks, namely, that the dynamics is Hamiltonian, governed by a variational principle. This is the case of ideal MHD, for example [1] Sect. 6.5.

Suppose the Lagrangian is $L(y, \dot{y}, t)$, where y is restricted to the modal representation Eq. (1). Rotational invariance suggests taking the Lagrangian L, expressed in terms of a, as

$$2L = |\dot{a}|^2 + \mu|a|^2 + \sigma|a|^4, \tag{6}$$

where μ and σ are real parameters. To carry out the variation with y, it is convenient again to write $a = r\exp(i\xi)$, and treat r and ξ as independent variables.

The Lagrangian becomes

$$2L = \dot{r}^2 + r^2\dot{\xi}^2 + \mu r^2 + \sigma r^4, \tag{7}$$

whence the variational equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \ddot{r} - r\dot{\xi}^2 - \mu r - 2\sigma r^3 = 0, \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}}\right) - \frac{\partial L}{\partial \xi} = \frac{d}{dt}(r^2\dot{\xi}) = 0. \tag{9}$$

Equation (9) implies $\dot{\xi} = C/r^2$, for Casimir constant C. The reflectionally symmetric case $C = 0$ is easy to understand. It represents standing waves with amplitude obeying the equation

$$\ddot{r} = \mu r + 2\sigma r^3. \tag{10}$$

Compared to Eq. (4), this admits quite different, oscillatory dynamics because the time derivative has changed to second order. Equation (10) is easiest to understand by considering motion in the potential corresponding to its first integral, although its solution may also be given explicitly in terms of Jacobi elliptic functions.

It seems reasonable to assume that initially r is small $= \epsilon$ at the onset of instability. For the case where $\mu > 0$, $\sigma < 0$, r is then forced to grow slowly while its amplitude is small. However, it will grow at an ever-increasing rate, until the r^3 term kicks in when $r = \mathcal{O}(\sqrt{(\mu/\sigma)})$ and just as rapidly returns it to a low level. In other words the system will be generically bursty, with its solutions suddenly rising up by a factor $\mathcal{O}(1/\epsilon)$.

Interaction with an axisymmetric mode, representing the tokamak equilibrium configuration, will, in order to satisfy equivariance, be via a term proportional to $|a|^2 = r^2$. Suppose the axisymmetric mode z is governed by dissipative dynamics, then the simplest bifurcation model is the fold, which with the interaction term added, is

$$\dot{z} = \alpha - \beta z^2 - r^2. \tag{11}$$

As Fig. 1 shows, the equilibrium mode exhibits saturated cyclic behavior, the crashes corresponding to the bursts of the nonaxisymmetric mode. The choice of coefficients is reasonable in that when z is rescaled to be of order unity at saturation, Eq. (11) takes the weakly coupled form $d\zeta/dt = 1 - \zeta^2 - \epsilon_1 r^2$, with $\epsilon_1 = \mathcal{O}(10^{-2})$.

There is also a question concerning how finite dissipation affects these ideal models. However, there is good evidence that oscillatory behavior persists at least at higher amplitude, although small amplitude nonreversing oscillations may damp. The degenerate, *symmetric* Takens-Bogdanov bifurcation, which includes terms representing dissipation, contains Eq. (10) in its unfolding [[13] Sect. 7.3], and exhibits oscillation.

The sawtooth mode is a possible candidate for identification with the Taylor cells since it is almost as ubiquitous in tokamaks as the Taylor cell is in T-C flow, and both

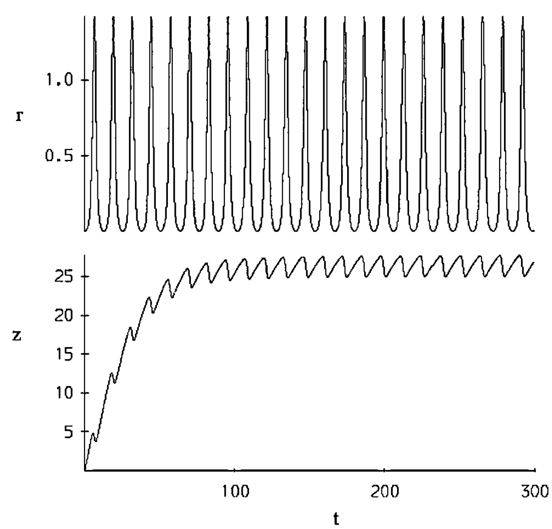

FIG. 1. Time series plots of solutions to the system of Eq. (10) coupled to Eq. (11). Parameters $\mu = 1$, $2\sigma = -1$, $\alpha = 1$, $\beta = 0.001$, initial values $r = 0.01$ and $z = 0$.

are nontraveling waves. The fact that the sawtooth continues to oscillate in time is accounted for by the relative smallness of dissipative effects in the tokamaks compared to T-C flow. The tentative identification of ''kinetic'' tearing modes with T-C traveling waves is natural because, as well as possessing the same qualitative behavior, neither normally occurs until there is already a different wave pattern present.

Conclusions.—Using only relatively simple equivariant bifurcation theory, this Letter has reproduced the principal qualitative features of Ohmically heated tokamak discharges, namely, (i) saturated traveling waves in a generic dissipative model. (ii) Bursty and sawtoothing behavior in a generic model with an ideal symmetry-breaking mode.

This has important theoretical ramifications, for example (2) means that the fact that a physical model exhibits bursty or sawtoothing behavior is no sure guarantee that it fully explains sawteeth or ELMs: any model obeying the symmetry constraints will exhibit such behavior, which is generic to ideal axisymmetric models. On a more positive note, however, these results support the contention that ODEs of the types discussed can be used to fit to experimental data using, say, methods from Ref. [14], whence they should produce quantitative information regarding the nonlinear terms. This information should be useful to compare with physical theories and also possibly in the devising of feedback control strategies to suppress mode growth.

Further work needs to be pursued in parallel with any application to experimental analysis. First, it is likely that in key regions of operating space, two or possibly even more different modes are simultaneously close to instability. Hence, higher order, degenerate bifurcation theories of the kind described in Ref. [8] need to be developed for symmetric systems. Second, it would be sensible to look systematically at the effect of introducing small amounts of dissipation into the bursty model, to include small symmetry-breaking terms [15] in order to investigate, e.g., the effects of magnetic field control coils, and to add small random terms [16] to model small-scale turbulence.

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