Anomalous Refraction of Light Colors by a Metamaterial Prism

Mário G. Silveirinha*

Department of Electrical Engineering–Instituto de Telecomunicações, University of Coimbra, 3030 Coimbra, Portugal
(Received 7 January 2009; revised manuscript received 26 March 2009; published 14 May 2009)

A prism of glass separates white light into its spectral components in such a manner that colors associated with shorter wavelengths are more refracted than the colors associated with longer wavelengths. Here, we demonstrate that this property is not universal, and that a lossless metamaterial prism with a suitable microstructure may enable a broadband regime of anomalous dispersion, where the spectral components of light are separated in an unconventional way, so that ‘‘violet light’’ is less refracted than ‘‘red light.’’ This phenomenon is fundamentally different from conventional anomalous dispersion effects, which are invariably accompanied by significant loss and are typically very narrow band.

DOI: 10.1103/PhysRevLett.102.193903 PACS numbers: 42.70.Qs, 41.20.Jb, 78.20.Ci, 78.66.Sq

The separation of the colors of light by a glass prism is one of the most remarkable optical phenomena. Such a beautiful effect is a consequence of the frequency dispersion of the index of refraction of glass, which makes spectral components associated with different colors be refracted in different directions. Similarly, the dazzling profusion of colors observed in rainbows is the result of the wavelength dependence of the refractive index of water droplets. In a transparent medium, such that the absorption is negligible in a given range of frequencies, the electric permittivity ε must be an increasing function of frequency [1]. This property stems from the principle of causality and from one of the Kramers-Kronig formulas for $\varepsilon(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$:

$$\varepsilon'(\omega) = 1 + \frac{1}{\pi}\text{P.V.}\int_{-\infty}^{+\infty} \frac{\varepsilon''(u)}{u-\omega}du, \quad (1)$$

where P.V. stands for the principal value of the integral [1]. When the material is transparent near the frequency ω of interest, i.e., $\varepsilon''(\omega)$ is negligibly small, it is possible to differentiate the integral in the usual way because in practice $u = \omega$ does not lie in the integration region. Hence, we have that [1]

$$\frac{\partial\varepsilon'}{\partial\omega} = \frac{4\omega}{\pi}\int_{0}^{+\infty} \frac{u\varepsilon''(u)}{(u^2-\omega^2)^2}du > 0, \quad (2)$$

which shows that indeed the refractive index $n = \sqrt{\varepsilon(\omega)}$ increases with frequency (for simplicity, it is assumed that $\mu = 1$), and consequently that the spectral components of light associated with shorter wavelengths (‘‘violet’’ light) are more refracted by a glass prism than the components associated with longer wavelengths (‘‘red’’ light), as illustrated in Fig. 1(a). Hence, it seems that, independent of the material of the prism, the palette of refracted colors will always follow the same pattern, showing red as the leftmost color and violet as the rightmost color. Indeed, it is well-known that a regime of anomalous dispersion, where the refracting index decreases with frequency, requires the presence of loss and is typically very narrow band [2,3].

In recent years, there has been a great interest in novel artificially engineered materials, known as metamaterials. Such structured materials are typically formed by dielectric or metallic inclusions, whose shape and size are rationally designed in order to tailor the electromagnetic properties of the composite material. Metamaterials may enable an unprecedented control of the propagation of electromagnetic waves, and effects like negative refraction and subwavelength imaging [4,5].

Here, we demonstrate that, notwithstanding the fundamental physical constraints that impose that in a transparent material $\partial\varepsilon/\partial\omega > 0$, it may be possible to design a metamaterial prism with a suitable microstructure such that it separates the colors of light in an anomalous way, and the palette of refracted colors is reversed [Fig. 1(b)]. To this end, we need to step beyond the frontier of usual local materials, and consider materials with a nonlocal (spatially dispersive) response [1,6]. A spatially dispersive material is characterized by the fact that the electric dipole moment

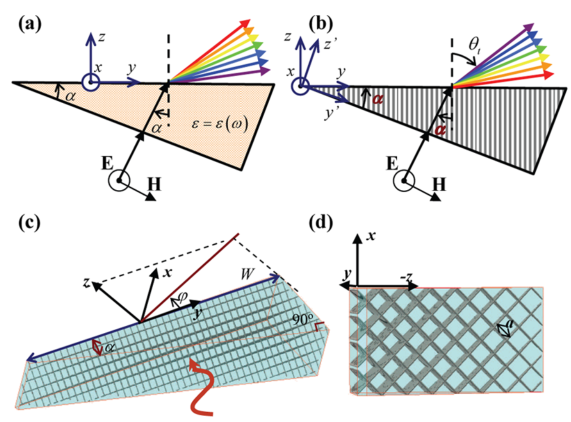

FIG. 1 (color online). (a) Refraction of light by a conventional dielectric prism. (b) Refraction of light by the metamaterial prism proposed in this Letter. Notice that the palette of refracted colors is reversed. (c) and (d) Perspective views of the metamaterial prism. The metamaterial is formed by nonconnected crossed metallic wires that lie in planes normal to the y direction.

acquired by a basic particle cannot be written exclusively in terms of the local electric field in a small neighborhood of the particle, but ultimately may depend on the electric field distribution in the whole crystal. It is well-known that the plane waves in an unbounded spatially dispersive material are characterized by a dielectric function of the form $\bar{\bar{\varepsilon}}(\omega, \mathbf{k})$, where $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector. For simplicity, let us consider the particular case where the wave propagates in the yoz plane, i.e., $k_x = 0$, and the electric field and the electric displacement are collinear and polarized along the x direction. In these conditions the wave propagation can be completely described by the ε_{xx} component of the dielectric function, which is denoted hereafter by $\varepsilon(\omega, \mathbf{k})$. As in a local material, as a consequence of causality, for each fixed $\mathbf{k}$ the dielectric function $\varepsilon(\omega, \mathbf{k})$ also verifies the Kramers-Kronig formulas, and, in particular, in a transparency window it verifies Eq. (2) [6] (p. 53; the material is assumed nongyrotropic). However, the key point is that unlike in a local material, the property $\partial\varepsilon/\partial\omega > 0$ does not imply that the refractive index $n = \sqrt{\varepsilon(\omega, \mathbf{k})}$ is an increasing function of frequency. In fact, for a plane wave the wave vector $\mathbf{k}$ must verify the dispersion characteristic $k_y^2 + k_z^2 = (\omega/c)^2\varepsilon(\omega, \mathbf{k})$, and should be regarded a function of frequency. Hence, even though $\partial\varepsilon/\partial\omega > 0$, the total derivative $\frac{d\varepsilon}{d\omega} = \frac{\partial\varepsilon}{\partial\omega} + \nabla_k\varepsilon \cdot \frac{d\mathbf{k}}{d\omega}$ does not need to be positive, and the index of refraction may possibly be a decreasing function of frequency.

In order to demonstrate this possibility, we consider an artificial material formed by two nonconnected arrays of parallel wires ("double-wire medium") [7]. Each array of parallel wires is arranged in a square lattice with lattice constant a. The two arrays of wires are mutually orthogonal and lie in planes parallel to the xoz plane. The distance between adjacent perpendicular wires is $a/2$. The metallic wires are tilted by $\pm 45^\circ$ with respect to the xoy plane [Figs. 1(c) and 1(d)]. For simplicity, it is supposed that the wires stand in air. It is known that the considered material is strongly spatially dispersive, and for $k_x = 0$ and an electric field polarized along the x direction, it is characterized by the dielectric function

$$\varepsilon(\omega, k_z) = 1 - \frac{\beta_p^2}{(\omega/c)^2 - k_z^2/2}, \tag{3}$$

where $\beta_p = \{2\pi/[\ln(a/2\pi r_w) + 0.5275]\}^{1/2}/a$ is the plasma wave number, and r_w is the radius of the wires [7]. For simplicity, it is assumed here that the wires are perfect electric conductors. The effect of metallic loss can be easily accounted for using the formulas of Ref. [7], and is negligible provided the skin depth of the metal is smaller than the radius of the wires. It was demonstrated in Ref. [7] that, unlike what could be expected based on the conventional model that describes wire media as an artificial plasma, the metamaterial supports a propagating mode characterized by a large positive index of refraction for arbitrarily long wavelengths. Such a property may enable the realization of ultrasubwavelength waveguides, as experimentally verified in Ref. [8], and subwavelength imaging [9]. Here, we prove that a prism of the metamaterial separates the colors of light in an unusual manner, reversing the palette of refracted colors.

Consider a plane wave propagating along the z direction in the unbounded metamaterial. From Eq. (3) it is clear that $\frac{\partial\varepsilon}{\partial k_z} = -\frac{\partial\varepsilon}{\partial\omega}\frac{k_z}{\omega}\frac{c^2}{2}$. Thus, the total derivative of the permittivity seen by the plane wave with respect to frequency may be written as $\frac{d\varepsilon}{d\omega} = \frac{\partial\varepsilon}{\partial\omega}(1 - \frac{1}{2}\frac{c^2}{v_p v_g})$, where $v_p = \omega/k_z$ and $v_g = d\omega/dk_z$ are the phase and group velocities, respectively. Hence, even though $\partial\varepsilon/\partial\omega > 0$, when $0 < v_p v_g < c^2/2$ the refractive index may, indeed, decrease with frequency. In fact, substituting the dielectric function (3) into the dispersion characteristic of plane waves, it is readily found that the index of refraction (for propagation along z or quasiparallel to z) is [7]

$$n = \sqrt{\frac{3}{2} + \frac{1}{2}\sqrt{1 + 8\left(\frac{\beta_p c}{\omega}\right)^2}}, \tag{4}$$

which is clearly a decreasing function of frequency, because β_p depends exclusively on the geometry of the metamaterial. This exciting property suggests that a prism of the considered metamaterial may refract the spectral components of light in an unconventional manner. Indeed, since the component of the wave vector parallel to the interface must be preserved, it is clear that for normal incidence the wave vector in the metamaterial must be collinear with the wave vector associated with the incident wave. For similar reasons, the angle of refraction θ_t is such that $\sin\theta_t = n\sin\alpha$, where α is the angle between the input and output interfaces of the prism [10]. Thus, since n decreases with frequency, the angle of refraction θ_t has exactly the same property, and consequently colors of light associated with shorter wavelengths will be less refracted by the prism. It is interesting to mention that, similar to the phase velocity, the group velocity also increases with frequency, and is always less than c.

To illustrate the suggested potentials we plot in the inset of Fig. 2 the angle of transmission, θ_t, as a function of the normalized frequency and for different values of α. The metamaterial is formed by wires with radius $r_w = 0.05a$, where a is the lattice constant. It is clear from Eq. (4) that for sufficiently low frequencies the wave suffers total internal reflection. The onset of transmission ($\theta_t = 90^\circ$) occurs at the frequency ω_c such that $n(\omega_c) = \csc\alpha$. As shown in Fig. 2, for frequencies larger than ω_c the angle θ_t decreases monotonically. Notice that in the spectral region of interest $\omega a/c \ll 1$; i.e., the lattice constant is much smaller than the wavelength, and thus the metamaterial can indeed be described using the homogenization model.

In order to further characterize the anomalous properties of the metamaterial, next we investigate the refraction of a Gaussian beam by the metamaterial prism. For simplicity, it is assumed that the prism is periodic along the x direction and that the incoming electromagnetic field is independent

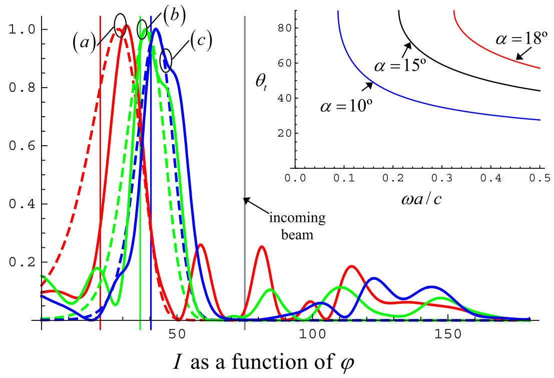

FIG. 2 (color online). Normalized radiation intensity I of the refracted beam (calculated at a distance $R = 30.0\lambda_0$ from the output interface) as a function of the elevation angle φ, for a metamaterial prism with $\alpha = 15^\circ$ and (a) $\omega = 0.25c/a$, (b) $\omega = 0.35c/a$, and (c) $\omega = 0.40c/a$. The dashed lines correspond to the results obtained using physical optics [Eq. (6)], whereas the solid lines were obtained by solving Maxwell's equations using the periodic MoM. The vertical grid lines mark the theoretical angle of transmission θ_t for each of the considered cases. The inset shows θ_t as a function of normalized frequency and different values of α.

of x. The output interface of the prism is defined by $0 < y < W$, where W is the aperture of the prism. The incident electric field is polarized along x, and is such that

$$E_x^{\rm inc}(y', z') = \frac{E_0 e^{ik_0 z'}}{\sqrt{1 + 2iz'/k_0 w_0^2}} e^{-(y'-y'_0)^2/[w_0^2(1+2iz'/k_0 w_0^2)]}, \tag{5}$$

where y' and z' form a system of coordinates attached to the input interface (see Fig. 1), $y'_0 = W/2\cos\alpha$ is the coordinate of the central point of the input interface, $k_0 = \omega/c$ is the free-space wave number, and w_0 defines the beam waist. The electric field at the output interface (after propagation through the prism) may be estimated as $E_x^{\rm out}(y) = E_x^{\rm inc}(y\cos\alpha, 0)T(\omega; L_y)$, where $L_y = y\sin\alpha$ is the distance traveled by the wave inside the lens, and $T(\omega; L)$ is the transmission coefficient for plane wave incidence (along the normal direction) in a metamaterial slab with thickness L (see the inset of Fig. 3). The transmission coefficient can be evaluated using formula (5) of Ref. [9].

To give an idea of how T varies along the output interface of the prism, in Fig. 3 we plot the amplitude and phase of T as a function of L/a for the normalized frequency $\omega a/c = 0.22$. This frequency is slightly larger than the frequency ω_c that defines the onset of transmission through a prism with $\alpha = 15^\circ$, a configuration which will be studied in detail below. The solid lines correspond to the exact results obtained by solving Maxwell's equations using the periodic method of moments (MoM), whereas the discrete symbols were obtained using the homogenization model developed in Ref. [9], using $\beta_p = 1.76/a$ (this is slightly

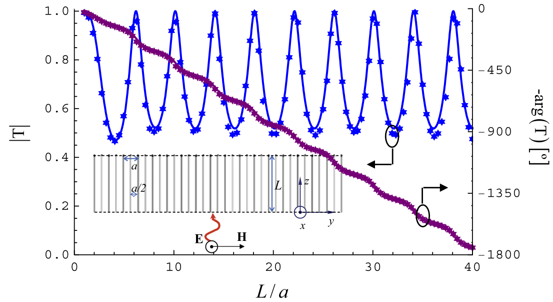

FIG. 3 (color online). Amplitude and phase of the transmission coefficient T of a planar metamaterial slab, as a function of the thickness of the slab L. The slab is illuminated by a plane wave propagating along the normal direction. The normalized frequency of operation is $\omega a/c = 0.22$, and the radius of the wires is $r_w = 0.05a$. Solid lines: full wave calculation with the periodic MoM. Discrete symbols: results obtained using the homogenization model developed in Ref. [9]. The inset represents the geometry of the problem. The wires are tilted by $\pm 45^\circ$ with respect to the interfaces, and lie in planes normal to the y direction.

smaller than the theoretical value $1.93/a$). Again, we emphasize that T is calculated by solving the scattering problem associated with a planar metamaterial slab. As explained in the previous paragraph, it is expected that if α is not too large, T can be used as well to estimate the field at the output interface of the prism. As could be anticipated, due to interference effects caused by the wave reflections at the slab interfaces, the amplitude of T oscillates nearly periodically as the thickness of the metamaterial is varied.

The effects of wave diffraction after transmission through the metamaterial prism can be estimated using the physical optics approximation

$$E_x(y, z) \approx \int_0^W E_x^{\rm out}(\bar{y})\left(-ik_0 \cos\theta_t \Phi_0 - \frac{\partial \Phi_0}{\partial z}\right) d\bar{y}, \tag{6}$$

where $\Phi_0 = \frac{i}{4} H_0^{(1)}(k_0\sqrt{(y-\bar{y})^2 + z^2})$, and $H_0^{(1)}$ is the Hankel function of first kind and order zero. We have used the above formalism to characterize the refracted Gaussian beam for a metamaterial prism formed by 297 planes of metallic wires spaced by $a/2$ ($W = 148a$), and with $\alpha = 15^\circ$. The radius of the wires is $r_w = 0.05a$. The beam waist of the incoming wave was taken equal to $2w_0 = 4.0\lambda_0$. In Fig. 2 we plot the radiation intensity, $I \sim |E_x|^2$, in the far field of the metamaterial prism (dashed lines) as a function of the elevation angle φ [see Fig. 1(c)] for different frequencies of operation. Consistently with the variation of θ_t with frequency (inset of Fig. 2), it is seen that as the frequency increases from $\omega a/c = 0.25$ up to $\omega a/c = 0.40$, the main lobe of the refracted beam approaches the direction normal to the prism; i.e., the maximum of the radiation intensity occurs for a larger value of φ, confirming that the metamaterial prism separates the spectral components of light in an anomalous manner. The

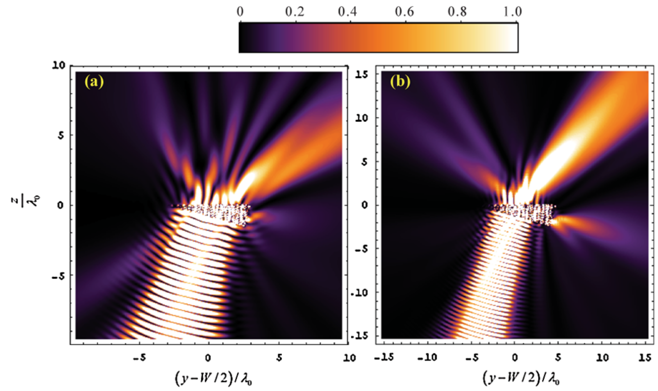

FIG. 4 (color online). Normalized $|E_x|^2$ (which is roughly proportional to the beam intensity) in the vicinity of the prism, calculated by solving Maxwell's equations using the periodic MoM. Panel (a): $\omega a/c = 0.25$. Panel (b): $\omega a/c = 0.40$.

direction of maximum radiation intensity is not exactly coincident with the theoretical value (marked by the vertical grid lines in Fig. 2) due to the effect of diffraction. Clearly, the effects of diffraction are more important for low frequencies, not only because the prism is electrically smaller but also because the angle of transmission θ_t is larger. In this example, the prism aperture is $W = 5.9\lambda_0$ at $\omega a/c = 0.25$, and increases to $W = 9.4\lambda_0$ at $\omega a/c = 0.40$. Naturally, for a prism with a larger aperture the effect of diffraction will be less significant.

To demonstrate in a conclusive manner the extraordinary properties of the metamaterial prism, we have solved Maxwell's equations using the MoM. The MoM takes into account all the details of the microstructure of the metamaterial prism, and gives the exact solution of the problem, apart from some small inevitable "numerical noise." To simplify the numerical modeling it is still assumed that the metamaterial prism is periodic along the x direction. The dimensions of the metamaterial prism are the same as before. The calculated (full wave) radiation intensity associated with the refracted beam is depicted in Fig. 2 (solid lines). It can be recognized that the full wave results are qualitatively similar to the results obtained using the physical optics approximation, and clearly show that the longer wavelengths are more refracted by the metamaterial prism than the shorter wavelengths. Figure 4 shows a density plot of $|E_x|^2$ (which is roughly proportional to the beam intensity) in the vicinity of the prism for $\omega a/c = 0.25$ and $\omega a/c = 0.40$. Unlike a conventional glass prism, it is seen that the metamaterial prism bends more the spectral component of light with the larger λ_0. Some fluctuations of the transmitted field may be noticed, especially in Fig. 4(a). These side lobe diffraction peaks occur due to the finite size of both the prism aperture and the beam waist, which leads to the excitation of guided modes at the edges of the prism [7–9]. The guided modes propagate strongly attached to the prism interface and form a standing wave due to reflection at the prism vertices, originating some oscillations in the near-field diagram. It is expected that for a prism with an electrically larger aperture W these diffraction related effects will be less important, as supported by the results of Fig. 4(b).

In conclusion, it was demonstrated that, notwithstanding the very strict physical principles that impose that in a transparent material $\partial\varepsilon/\partial\omega > 0$, the index of refraction of a spatially dispersive metamaterial formed by crossed metallic wires may decrease with frequency in a very broad spectral range. Such a remarkable property is absolutely unique and cannot be obtained with conventional local materials [2,3], which would require the presence of significant loss. It was demonstrated theoretically and with full wave numerical simulations that this property enables a metamaterial prism to reverse the palette of the refracted colors. The proposed material may be fabricated at microwaves using the technique reported in [8]. Even though at optical frequencies the response of metals is dominated by plasmonic effects and the real part of the permittivity is relatively low, the reported phenomenon may still be observed because of the very strong nonlocal response induced by the topology of the metamaterial (this response is comparatively much stronger than in an array of parallel wires, which may behave as a local material in the optical regime [11]).

This work is supported in part by Fundação para a Ciência e a Tecnologia, Grant No. PTDC/EEA-TEL/100245/2008.

*mario.silveirinha@co.it.pt

[1] L. D. Landau, E. Lifshitz, and L. Pitaevskii, *Electrodynamics of Continuous Media* (Elsevier Butterworth-Heinemann, Oxford, 2004), 2nd ed.
[2] M. D. Stenner, D. J. Gauthier, and M. A. Neifeld, Nature (London) **425**, 695 (2003).
[3] S. Chu and S. Wong, Phys. Rev. Lett. **48**, 738 (1982).
[4] J. Valentine, S. Zhang, T. Zentgraf, E. Ulin-Avila, D. A. Genov, G. Bartal, and X. Zhang, Nature (London) **455**, 376 (2008).
[5] J. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).
[6] V. M. Agranovich and V. Ginzburg, *Spatial Dispersion in Crystal Optics and the Theory of Excitons* (Wiley-Interscience, New York, 1966).
[7] M. G. Silveirinha and C. A. Fernandes, Phys. Rev. B **78**, 033108 (2008).
[8] M. G. Silveirinha, C. A. Fernandes, J. R. Costa, and C. R. Medeiros, Appl. Phys. Lett. **93**, 174 103 (2008).
[9] M. G. Silveirinha, C. A. Fernandes, and J. R. Costa, Phys. Rev. B **78**, 195121 (2008).
[10] Because of the nonlocal effects, the wire medium supports an additional evanescent wave, besides the wave described by the index of refraction (4) [7]. But since the angle of refraction θ_t depends uniquely on the phase variation at the output interface, and since the additional wave is exponentially attenuated in the metamaterial, it is clear that θ_t is determined exclusively by the propagating mode.
[11] J. Yao, Z. Liu, Y. Liu, Y. Wang, C. Sun, G. Bartal, A. M. Stacy, and X. Zhang, Science **321**, 930 (2008).