

## Scheme for Building a 't Hooft–Polyakov Monopole

Julian Sonner

*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139 USA*

David Tong

*Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, United Kingdom*

(Received 28 October 2008; published 11 May 2009)

We study a simple quantum mechanical model of a spinning particle moving on a sphere in the presence of a magnetic field. The system has two ground states. As the magnetic field is varied, the ground states mix through a non-Abelian Berry phase. We show that this Berry phase is the path ordered exponential of the smooth  $SU(2)$  't Hooft–Polyakov monopole. We further show that, by adjusting a potential on the sphere, the monopole becomes a Bogomol'nyi-Prasad-Sommerfield monopole and obeys the Bogomol'nyi equations.

DOI: 10.1103/PhysRevLett.102.191801

PACS numbers: 14.80.Hv, 03.65.Vf

*Introduction.*—In 1974, 't Hooft and Polyakov discovered a new solution of Yang-Mills-Higgs theories [1,2]. From afar, it looks like a Dirac magnetic monopole. However, the configuration is smooth, with the singularity at the origin of the Dirac monopole resolved by the non-Abelian gauge fields.

The spatial profile of the 't Hooft–Polyakov field configuration depends on the scalar potential for the adjoint-valued Higgs field  $\phi$ . Among these, one profile is rather special. This occurs when the potential vanishes and, as first shown by Prasad and Sommerfield [3], it is possible to find an exact solution. Later, Bogomol'nyi [4] showed that the non-Abelian field strength,  $\mathcal{F}_{\mu\nu}$ , for this configuration solves the simple, first order, differential equations

$$\mathcal{F}_{\mu\nu} = \epsilon_{\mu\nu\rho} \mathcal{D}_\rho \phi, \quad (1)$$

where  $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$  and  $\mathcal{D}_\rho \phi = \partial_\rho \phi + [\mathcal{A}_\rho, \phi]$ . Monopoles of this type are known as BPS, after the three authors named above. The subsequent discovery that these monopole play a special role in supersymmetric theories [5] has resulted in the title “BPS” being ascribed to almost anything associated to supersymmetry.

There is another, more abstract, situation in theoretical physics where the Dirac monopole arises. This is the Berry phase in quantum mechanics. Consider a spin 1/2 particle in a magnetic field  $\vec{B}$ . The Hamiltonian is given by

$$H = -\vec{B} \cdot \vec{\sigma} - |\vec{B}| 1_2, \quad (2)$$

where  $\vec{\sigma}$  are the Pauli matrices and  $1_2$  is the unit  $2 \times 2$  matrix, whose presence in the Hamiltonian simply ensures that the ground state energy of this two-state system is normalized to zero. We start in this ground state,  $|0\rangle$ . We then slowly rotate the magnetic field  $\vec{B}$  until, finally, we return to our initial setup. The adiabatic theorem in quantum mechanics tells us that the system remains in the

ground state and changes only by a phase. The question is, what is this phase? Since we have normalized the vacuum to zero energy, there is no dynamical contribution. Nonetheless, Berry showed that there is a geometrical phase which depends on the path  $\Gamma$  taken in the space of magnetic fields [6,7],

$$|0\rangle \rightarrow \exp\left(-i \oint_\Gamma \vec{A} \cdot d\vec{B}\right) |0\rangle. \quad (3)$$

The Abelian Berry connection  $\vec{A}$  is defined in terms of the dependence of the ground state on the magnetic field  $\vec{B}$ ,

$$\vec{A} = i\langle 0 | \frac{\partial}{\partial \vec{B}} | 0 \rangle. \quad (4)$$

Berry showed that, for the simple Hamiltonian (2), the connection (4) is that of the Dirac magnetic monopole:  $\vec{A} = \vec{A}^{\text{Dirac}}$ . One can form a  $U(1)$  field strength from the Berry connection in the usual way  $F_{\mu\nu} = \frac{\partial A_\mu}{\partial B_\nu} - \frac{\partial A_\nu}{\partial B_\mu}$ . This takes the radial, Dirac monopole form

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho} \frac{B_\rho}{B^3}. \quad (5)$$

Note that there is a potential for confusion here, because  $\epsilon_{\mu\nu\rho} F_{\mu\nu}$  is an abstract magnetic monopole over the space of real magnetic fields  $\vec{B}$ . The field strength  $F_{\mu\nu}$  has a singularity at the origin. This is nothing to be afraid of: it simply reflects the fact that the excited state and the ground state become degenerate at  $\vec{B} = 0$ . Indeed, the very existence of the Berry phase can be traced to this degenerate point in parameter space.

In this Letter, we ask whether the smooth non-Abelian 't Hooft–Polyakov monopole can appear as a Berry connection in simple quantum mechanical systems. The answer, as we shall see, is yes. The concept of the non-Abelian Berry connection was introduced by Wilczek and Zee [8]. This occurs if a system has degenerate eigen-

states for all values of the parameters  $\vec{B}$ . If there are  $N$  degenerate states  $|a\rangle$ ,  $a = 1, \dots, N$ , then after a cyclic and adiabatic tour through the space of parameters, the system will undergo a  $U(N)$  rotation,

$$|a\rangle \rightarrow P \exp\left(-i \oint_{\Gamma} \vec{\mathcal{A}}_{ab} \cdot d\vec{B}\right) |b\rangle, \quad (6)$$

where the  $u(N)$  valued Berry connection is defined by

$$\vec{\mathcal{A}}_{ab} = i \langle b | \frac{\partial}{\partial \vec{B}} | a \rangle. \quad (7)$$

To build an  $SU(2)$  't Hooft–Polyakov monopole as a Berry connection, we need to construct a Hamiltonian with two degenerate ground states for all values of the parameters  $\vec{B}$ . Moreover, since the 't Hooft–Polyakov monopole is smooth, our system should generate a topologically non-trivial Berry connection without any further degeneracies occurring in parameter space. (Non-Abelian monopoles have arisen previously in the context of Berry phases [9,10]. However, in both of these papers the configuration is not smooth, with a singularity at the origin resulting from an extra degeneracy of states.)

In this Letter, we show that these conditions arise for a spin 1/2 particle moving on a sphere  $S^2$  in the presence of a magnetic field  $\vec{B}$ . As  $\vec{B}$  is varied, the mixing (6) between the ground states is governed by a 't Hooft–Polyakov monopole. Moreover, we show that by including a potential over  $S^2$ , the monopole takes the BPS form, and the  $SU(2)$  Berry connection satisfies the Bogomol'nyi equation (1).

*Quantum mechanics and monopoles.*—In this section we introduce two simple quantum mechanical systems with degenerate ground states. Both of these have a non-Abelian  $SU(2)$  Berry phase described by a 't Hooft–Polyakov monopole. For the first, we have only an implicit description of the profile of the monopole. However, a small modification of this system allows us to solve for the Berry phase exactly and we find the BPS monopole satisfying (1).

Consider a neutral, spin 1/2 particle moving on a sphere  $S^2$  in the presence of a magnetic field. We introduce the usual polar coordinates  $\theta \in [0, \pi)$  and  $\phi \in [0, 2\pi)$  on the sphere. The magnetic field varies over the sphere and is given by  $\vec{B} \cos\theta$  where  $\vec{B}$  is a constant vector. The Hamiltonian is

$$H = -\frac{\hbar^2}{2m} \Delta_{12} - \hbar \vec{B} \cdot \vec{\sigma} \cos\theta. \quad (8)$$

The operator  $\Delta$  is the Laplacian on the unit  $S^2$

$$\Delta = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}. \quad (9)$$

The coordinate  $\phi$  is cyclic. This ensures that the ground states of the system do not depend on  $\phi$ .

The Hamiltonian enjoys a  $Z_2$  symmetry,

$$\vec{B} \rightarrow -\vec{B}, \quad \theta \rightarrow \pi - \theta. \quad (10)$$

The sign flip of the magnetic field acts on the Hilbert space by exchanging spin-up and spin-down states,  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , defined to be the two normalized eigenvectors of  $\vec{B} \cdot \vec{\sigma}$  with eigenvalues  $+|B|$  and  $-|B|$ , respectively.

The  $Z_2$  symmetry guarantees the existence of two ground states for all values of  $\vec{B}$ . For  $\vec{B} \neq 0$ , the spin-up state is localized near  $\theta = 0$ , while the spin-down state is localized near  $\theta = \pi$ . When  $\vec{B} = 0$ , both ground states are smeared uniformly over the sphere. However, in contrast to the Hamiltonian (2), there is no extra degeneracy of the ground states when  $\vec{B} = 0$ . For arbitrary values of  $\vec{B}$ , the two, normalized, ground states are a combination of the spin states and a spatial wave function,  $\psi(\cos\theta; B)$ , which depends on the magnitude  $B = |\vec{B}|$ ,

$$|1\rangle = \psi(\cos\theta; B) |\uparrow\rangle, \quad |2\rangle = \psi(-\cos\theta; B) |\downarrow\rangle. \quad (11)$$

Writing  $x = \cos\theta$ , the spatial wave function  $\psi(x; B)$  satisfies the Schrödinger equation,

$$-\frac{\hbar^2}{2m} (1-x^2) \psi'' + \frac{\hbar^2}{2m} x \psi' - \hbar B x \psi = E_0 \psi, \quad (12)$$

with  $\psi' = d\psi/dx$ , and  $E_0$  the ground state energy.

We now compute the Berry phase for this quantum mechanical system. The system is prepared in one of the ground states before the magnetic field  $\vec{B}$  is adiabatically varied, traversing a closed loop in parameter space. At the end of this tour, the ground state has undergone a  $U(2)$  rotation, defined, as in (6), by the path ordered exponential of the Berry connection,

$$\vec{\mathcal{A}}_{ab} = i \langle b | \frac{\partial}{\partial \vec{B}} | a \rangle, \quad a, b = 1, 2. \quad (13)$$

To build some intuition, let us start with the diagonal components of the connection. Consider a large magnetic field  $B \gg \hbar/m$ , which localizes the spatial part of each wave function close to a pole, at  $\theta = 0$  or  $\theta = \pi$ . Here the ground state knows little about the rest of the sphere and sees an effective Hamiltonian of the form (2). This gives rise to a  $U(1)$  Berry connection which is equal to that of a Dirac monopole,  $\vec{\mathcal{A}}^{\text{Dirac}}$ . In fact, a simple computation reveals that the diagonal components are independent of the spatial wave functions for all values of  $B$ , and are given by

$$\begin{aligned} \vec{\mathcal{A}}_{11} &= \langle 1 | \frac{\partial}{\partial \vec{B}} | 1 \rangle = \langle \uparrow | \frac{\partial}{\partial \vec{B}} | \uparrow \rangle = \vec{\mathcal{A}}^{\text{Dirac}}, \\ \vec{\mathcal{A}}_{22} &= \langle 2 | \frac{\partial}{\partial \vec{B}} | 2 \rangle = \langle \downarrow | \frac{\partial}{\partial \vec{B}} | \downarrow \rangle = -\vec{\mathcal{A}}^{\text{Dirac}}. \end{aligned} \quad (14)$$

In contrast, the off-diagonal terms describe the tunneling between the two different spin states, and depend on the spatial wave function of the particle  $\psi$ . They are

$$\begin{aligned}\vec{A}_{21} &= \langle 2 | \frac{\partial}{\partial \vec{B}} | 1 \rangle = f(B) \langle \downarrow | \frac{\partial}{\partial \vec{B}} | \uparrow \rangle, \\ \vec{A}_{12} &= \langle 1 | \frac{\partial}{\partial \vec{B}} | 2 \rangle = f(B) \langle \uparrow | \frac{\partial}{\partial \vec{B}} | \downarrow \rangle,\end{aligned}\quad (15)$$

where the function  $f(B)$  is the overlap,

$$f(B) = 2\pi \int_0^\pi \sin\theta d\theta \psi^*(-\cos\theta; B) \psi(\cos\theta; B). \quad (16)$$

Without specific knowledge of the ground state wave function  $\psi$ , we are unable to compute explicitly the profile  $f(B)$  of the non-Abelian Berry monopole. However, on general grounds, we know that  $f(B) \rightarrow 0$  as  $B \rightarrow \infty$  since the two spatial wave functions are localized at antipodal points on the sphere. In the opposite limit,  $B = 0$ , the two spatial wave functions coincide and  $f(0) = 1$ .

The Dirac monopole connection  $\vec{A}^{\text{Dirac}}$  necessarily contains a singularity along a half-line, known as the Dirac string. In the present context, this arises because it is not possible to globally define a basis of spin states  $|\uparrow\rangle$  and  $|\downarrow\rangle$  for all values of  $\vec{B}$ . Therefore any explicit computation of the components of  $\vec{A}_{ab}$ , using the basis shown in (14) and (15), necessarily suffers from the Dirac string. However, there does exist a gauge in which the non-Abelian connection  $\vec{A}$  is free from the Dirac string. To demonstrate this, one must first choose a  $\vec{B}$  dependent basis for  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , then rotate  $\vec{A}$  using a suitable singular gauge transformation [10,11]. The result is the non-Abelian Berry connection which takes the rotationally covariant form,

$$\mathcal{A}_\mu = \epsilon_{\mu\nu\rho} \frac{B_\nu \sigma^\rho}{2B^2} [1 - f(B)]. \quad (17)$$

This is the connection of a 't Hooft–Polyakov monopole. Note, first, that it is an  $su(2)$  connection, rather than  $u(2)$ . Moreover, and most importantly, the asymptotic behavior of  $f(B)$  described above guarantees that, as  $B \rightarrow \infty$ , it reduces to the Dirac monopole for a  $U(1) \subset SU(2)$ . Yet the field strength is smooth at  $B = 0$ .

*The BPS monopole.*—Any deformation of the Hamiltonian (8) that preserves the vacuum degeneracy will again lead to a 't Hooft–Polyakov monopole with a different profile function  $f(B)$ . For example, we may add a spin-blind potential to the Hamiltonian. Something special happens for the potential given by

$$V(\theta) = \frac{1}{2} m B^2 \sin^2 \theta. \quad (18)$$

For this choice, the Schrödinger equation simplifies. The ground state energy is  $E_0 = 0$ , and it is a simple matter to find the exact wave functions. They are given by (11), with

$$\psi(\cos\theta; B) = \left( \frac{Bm/\hbar}{2\pi \sinh(2Bm/\hbar)} \right)^{1/2} e^{(Bm/\hbar)\cos\theta}. \quad (19)$$

Equations (16) and (17) then tell us the exact Berry connection for this system:

$$\mathcal{A}_\mu = \epsilon_{\mu\nu\rho} \frac{B_\nu \sigma^\rho}{2B^2} \left( 1 - \frac{2Bm/\hbar}{\sinh(2Bm/\hbar)} \right). \quad (20)$$

Remarkably, this is exactly the profile function of the BPS monopole satisfying (1). We could ask whether the adjoint-valued Higgs field,  $\phi$ , also has a counterpart in our quantum mechanics. Indeed, it is given by the  $su(2)$  valued expectation value,

$$\phi_{ab} = \frac{2m}{\hbar} \langle b | \cos\theta | a \rangle. \quad (21)$$

Using the exact ground state (19), and after performing the gauge transformation to the rotationally covariant gauge described above, we find the scalar field profile

$$\phi = \frac{B_i \sigma^i}{B^2} \left( \frac{2Bm}{\hbar} \coth\left(\frac{2Bm}{\hbar}\right) - 1 \right), \quad (22)$$

which is precisely the form of the Higgs field for the BPS monopole solution of  $SU(2)$  Yang-Mills Higgs theory (1). The magnetic field  $\vec{B}$  plays the role of the spatial position, while the analog of the Higgs expectation value is  $2m/\hbar$ .

*Discussion.*—The fact that one can write an equation, such as (1), to describe the Berry connection is intriguing. Typically, the only way to compute the Berry connection is through the direct definition (7), but to do this one first needs to compute the exact ground states as a function of the parameters. If the Berry connection can be shown to obey an equation—for example, of the form (1)—then one can circumvent this step. In fact, this shortcut is known to happen in supersymmetric quantum mechanics. This was first shown some years ago [12–14] for quantum mechanical models where the Berry connection was shown to obey a formula known as the  $tt^*$  equation. More recently, other supersymmetric models have been exhibited for which it can be shown that the Berry connection satisfies the Bogomol'nyi equation (1), or generalizations thereof [15,16]. Interestingly, the Berry phase of supersymmetric quantum mechanics of this type has recently arisen in the context of black hole microstates [17].

Although the model described in this Letter is not supersymmetric, it may not surprise the reader to learn that, for the specific potential (18), it may be shown to be the truncation of a supersymmetric quantum mechanics [18]. The emergence of the Bogomol'nyi monopole equation thus, once again, nods towards the existence of an underlying supersymmetry.

In this Letter we have introduced a simple-quantum mechanical model of a particle moving on the sphere which gives rise to a smooth, non-Abelian Berry connection taking the form of a 't Hooft–Polyakov monopole. However, the periodic form of the Hamiltonian (8) makes it clear that it can also be given the interpretation of a particle moving in a one-dimensional potential lattice with an applied periodic magnetic field. Indeed, the exact ground states (19) are of the required Bloch form with vanishing crystal momentum. The monopole Berry phase derived in

this Letter then describes the response of the Bloch states to adiabatic changes in the magnetic field. It would be interesting to study the effects of this smooth non-Abelian Berry phase on the band structure of such  $1d$  crystals in more detail. Related systems, where the Berry phase depends on the reciprocal lattice, were previously studied in [19].

This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement No. DEFG02-05ER41360. J. S. is supported by NSF Grant No. PHY-0600465 and thanks the members of CTP at MIT, and especially Dan Freedman, for hospitality. D. T. is supported by the Royal Society.

- 
- [1] G. 't Hooft, Nucl. Phys. **B79**, 276 (1974).
  - [2] A.M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].
  - [3] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. **35**, 760 (1975).
  - [4] E. B. Bogomoln'yi, Sov. J. Nucl. Phys. **24**, 49 (1976).
  - [5] E. Witten and D. I. Olive, Phys. Lett. B **78**, 97 (1978).
  - [6] M. V. Berry, Proc. R. Soc. A **392**, 45 (1984).
  - [7] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).
  - [8] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).
  - [9] J. Moody, A. Shapere, and F. Wilczek, Phys. Rev. Lett. **56**, 893 (1986).
  - [10] C.-H. Chern, H.-D. Chen, C. Wu, J.-P. Hu, and S.-C. Zhang, Phys. Rev. B **69**, 214512 (2004).
  - [11] A. S. Goldhaber and W. P. Trower, Am. J. Phys. **58**, 429 (1990); J. Arafune, P. G. O. Freund, and C. J. Goebel, J. Math. Phys. (N.Y.) **16**, 433 (1975); Y. Ohnuki and S. Kitakado, J. Math. Phys. (N.Y.) **34**, 2827 (1993).
  - [12] S. Cecotti, L. Girardello, and A. Pasquinucci, Int. J. Mod. Phys. A **6**, 2427 (1991).
  - [13] S. Cecotti, Nucl. Phys. **B355**, 755 (1991).
  - [14] S. Cecotti and C. Vafa, Nucl. Phys. **B367**, 359 (1991).
  - [15] J. Sonner and D. Tong, J. High Energy Phys. 01 (2009) 063.
  - [16] C. Pedder, J. Sonner, and D. Tong, Phys. Rev. D **77**, 025009 (2008).
  - [17] J. de Boer, K. Papadodimas, and E. Verlinde, arXiv:0809.5062.
  - [18] J. Sonner and D. Tong, arXiv:0809.3783.
  - [19] Z. Fang *et al.*, Science **302**, 92 (2003).