Inflationary Universe with Anisotropic Hair

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We study an inflationary scenario with a vector field coupled with an inflaton field and show that the inflationary Universe is endowed with anisotropy for a wide range of coupling functions. This anisotropic inflation is a tracking solution where the energy density of the vector field follows that of the inflaton field irrespective of initial conditions. We find a universal relation between the anisotropy and a slow-roll parameter of inflation. Our finding has observational implications and gives a counterexample to the cosmic no-hair conjecture.

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Introduction.—Recent developments of precision cosmology have yielded a slight shift of an inflationary paradigm [1]. Before precision cosmology, zeroth order predictions of inflationary scenarios were sufficient. Indeed, curvature fluctuations had been supposed to be statistically homogeneous, isotropic, Gaussian and almost scale invariant. However, because of progress in observations, we are now forced to look at fine structures of fluctuations such as spectral tilt, non-Gaussianity, parity violation, and so on [2]. In fact, we need theoretical predictions at a percent level. Those precise predictions of inflationary scenarios will provide a clue to understand fundamental physics such as superstring theory when they are compared with observations.

In this Letter, we focus on a role of a vector field in the early Universe [3]. Of course, no one doubts existence of vector fields. At the same time, it is widely believed vector hair will disappear during the inflation conforming to the cosmic no-hair conjecture [4]. However, recently, it has been shown that anisotropic hair in the inflationary Universe can exist [5,6], although there may be perturbative instability in this specific realization [7]. Hence, it is worth seeking other models. At this point, we should recall that primordial magnetic fields are produced during inflation [8]. For example, the nonminimal kinetic term of vector fields in supergravity can be used to generate the primordial cosmological magnetic fields [9]. This fact suggests that we have a vector hair during inflation. Here, there is prejudice that the vector hair is negligibly small and it is legitimate to ignore the backreaction of magnetic fields to geometry. However, in the context of the precision cosmology, we should not neglect the backreaction if it is around a percent level [10]. Hence, it is important to quantify how small it is. Based on this observation, we study an inflationary scenario where the inflaton is coupled with the kinetic term of a massless vector field. Apparently, our model is free from instability. Interestingly, we find a tracking behavior of the energy density of the vector field. As a consequence, we show that there exist sizable vector hair quite generally. That yields a percent level anisotropic inflation.

It should be stressed that the presence of the vector hair in the early Universe breaks the rotational invariance and therefore provides various interesting phenomenological consequences [11]. Moreover, anisotropic inflation might give rise to a percent level correlation between primordial gravitational waves and cosmic microwave background radiations (CMB), which might be testable by CMB observations near future [6]. Therefore, "hairy inflation" is phenomenologically rich.

Basic equations.—We consider the following action for the gravitational field, the inflaton field ϕ and the vector field A_μ coupled with ϕ:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{1}$$

where g is the determinant of the metric, R is the Ricci scalar, $V(\phi)$ is the inflaton potential, $f(\phi)$ is the coupling function of the inflaton field to the vector one, respectively. The field strength of the vector field is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Thanks to the gauge invariance, we can choose the gauge $A_0 = 0$. Without loss of generality, we can take x axis in the direction of the vector. Hence, we take the homogeneous fields of the form $A_\mu = (0, A_x(t), 0, 0)$ and $\phi = \phi(t)$. Note that we have assumed the direction of the vector field does not change in time, for simplicity. This field configuration holds the plane symmetry in the plane perpendicular to the vector. Then, we take the metric to be

$$ds^2 = -dt^2 + e^{2\alpha(t)}[e^{-4\sigma(t)}dx^2 + e^{2\sigma(t)}(dy^2 + dz^2)], \tag{2}$$

where the cosmic time t is used. Here, e^α is an isotropic scale factor and σ represents a deviation from the isotropy. With above ansatz, one obtains the equation of motion for the vector field which is easily solved as

$$\dot{A}_x = f^{-2}(\phi)e^{-\alpha-4\sigma}p_A, \quad (3)$$

where an overdot denotes the derivative with respect to the cosmic time t and p_A denotes a constant of integration. Substituting (3) into other equations, we obtain basic equations

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{p_A^2}{2}f^{-2}(\phi)e^{-4\alpha-4\sigma}\right], \quad (4)$$

$$\ddot{\alpha} = -3\dot{\alpha}^2 + \kappa^2 V(\phi) + \frac{\kappa^2 p_A^2}{6}f^{-2}(\phi)e^{-4\alpha-4\sigma}, \quad (5)$$

$$\ddot{\sigma} = -3\dot{\alpha}\,\dot{\sigma} + \frac{\kappa^2 p_A^2}{3}f^{-2}(\phi)e^{-4\alpha-4\sigma}, \quad (6)$$

$$\ddot{\phi} = -3\dot{\alpha}\,\dot{\phi} - V'(\phi) + p_A^2 f^{-3}(\phi)f'(\phi)e^{-4\alpha-4\sigma}, \quad (7)$$

where a prime denotes the derivative with respect to ϕ.

From Eq. (4), we see the effective potential $V_{\rm eff} = V + p_A^2 f^{-2}e^{-4\alpha-4\sigma}/2$ determines the inflaton dynamics. As the second term is coming from the vector contribution, we refer it to the energy density of the vector. Let us check if inflation occurs in this model. Using Eqs. (4) and (5), the equation for the acceleration of the Universe is given by

$$\ddot{\alpha} + \dot{\alpha}^2 = -2\dot{\sigma}^2 - \frac{\kappa^2}{3}\dot{\phi}^2 + \frac{\kappa^2}{3}\left[V - \frac{p_A^2}{2}f^{-2}e^{-4\alpha-4\sigma}\right]. \quad (8)$$

We see that the potential energy of the inflaton needs to be dominant for the inflation to occur. Now, we assume the energy density of the vector can be negligible compared to that of the inflaton for the inflaton dynamics. Then, we examine when the anisotropy is not diluted during inflation. From Eq. (6), it is apparent that the fate of anisotropic expansion rate $\Sigma \equiv \dot{\sigma}$ depends on the behavior of coupling function $f(\phi)$. In the critical case $f(\phi) \propto e^{-2\alpha}$, the energy density of the vector field as a source term in Eq. (6) remains almost constant during the slow-roll inflation. Using slow-roll equations

$$\dot{\alpha}^2 = \frac{\kappa^2}{3}V(\phi), \qquad 3\dot{\alpha}\,\dot{\phi} = -V'(\phi), \quad (9)$$

we obtain $d\alpha/d\phi = \dot{\alpha}/\dot{\phi} = -\kappa^2 V(\phi)/V'(\phi)$. This can be easily integrated as $\alpha = -\kappa^2\int V/V' d\phi$. Here, we have absorbed a constant of integration into the definition of α. Thus, we obtain

$$f = e^{-2\alpha} = e^{2\kappa^2\int (V/V')d\phi}. \quad (10)$$

For the polynomial potential $V \propto \phi^n$, we have $f = e^{\kappa^2\phi^2/n}$. Given the critical case (10), we can parameterize the coupling function as [9]:

$$f = e^{2c\kappa^2\int (V/V')d\phi}, \quad (11)$$

where c is a parameter.

Naively, the energy density of the vector field grows during inflation when $c > 1$, which is the case we want to consider. It would not be possible to neglect the vector field in this case, and Eq. (9) would not be appropriate for discussing the inflation dynamics anymore. Let us see what happens if the vector field is not negligible.

Tracking anisotropic inflation.—To make the analysis concrete, we consider chaotic inflation with the potential $V(\phi) = m^2\phi^2/2$ ($n = 2$). For this potential, the coupling function becomes $f(\phi) = e^{c\kappa^2\phi^2/2}$. It is instructive to see what happens by solving Eqs. (4)–(7) numerically. In Fig. 1, we have shown the phase flow in $\phi - \dot{\phi}$ space where we can see two slow-roll phases, which indicates something different from the conventional inflation occurs. In Fig. 2, we have calculated the evolution of the anisotropy $\Sigma/H \equiv \dot{\sigma}/\dot{\alpha}$ for various parameters c under the initial conditions $\sqrt{c}\kappa\phi_i = 17$. As expected, all of solutions show a rapid growth of anisotropy in the first slow-roll phase. However, the growth of the anisotropy eventually stops at the order of a percent. Notice that this attractor like behavior is not so sensitive to a parameter c.

Now, we will give an analytic explanation of the numerical results and find a quite remarkable relation between the anisotropy and a slow-roll parameter of inflation.

As the energy density of the vector field should be subdominant during inflation, we can ignore σ in Eqs. (4), (5), and (7). However, in Eq. (6), all terms would be of the same order. Now, Eqs. (4) and (7) can be written as

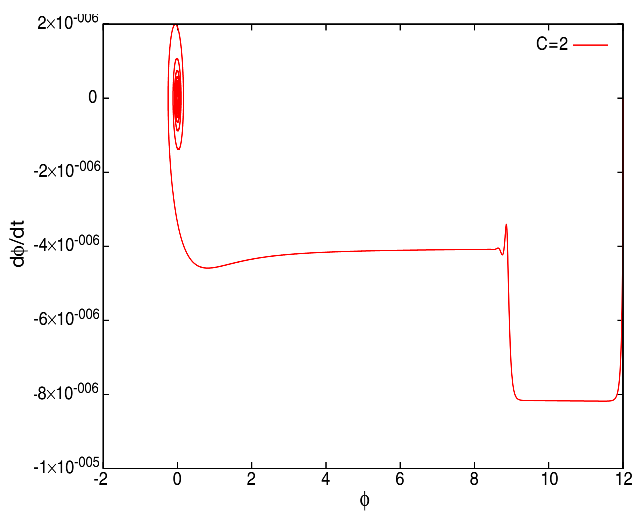

FIG. 1 (color online). Phase flow for ϕ is depicted. Here, we took the parameters $c = 2$ and $\kappa m = 10^{-5}$. We also put initial conditions $\phi_i = 12$ and $\dot{\phi}_i = 0$. There are two different slow-roll phases. The transition occurs around $\kappa\phi = 9$.

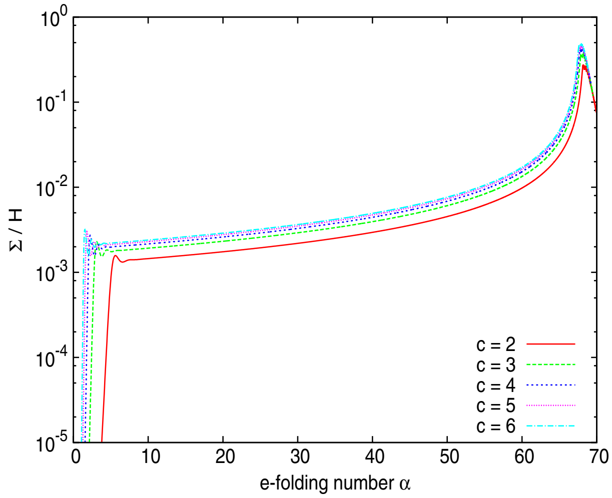

FIG. 2 (color online). Evolutions of the anisotropy Σ/H for various c are shown. One can see the attractorlike behavior of the anisotropy.

$$\dot{\alpha}^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}e^{-c\kappa^2\phi^2-4\alpha}p_A^2\right], \tag{12}$$

$$\ddot{\phi} = -3\dot{\alpha}\,\dot{\phi} - m^2\phi + c\kappa^2\phi e^{-c\kappa^2\phi^2-4\alpha}p_A^2. \tag{13}$$

Let us see how the energy density of the vector field works in these equations. When the effect of the vector field is comparable with that of the inflaton field as source terms in (13), we get the relation $c\kappa^2 p_A^2 e^{-c\kappa^2\phi^2-4\alpha} \sim m^2$. If we define the ratio of the energy density of the vector field $\rho_A \equiv p_A^2 e^{-c\kappa^2\phi^2-4\alpha}/2$ to that of the inflaton $\rho_\phi \equiv m^2\phi^2/2$ as

$$\mathcal{R} \equiv \frac{\rho_A}{\rho_\phi} = \frac{p_A^2 e^{-c\kappa^2\phi^2-4\alpha}}{m^2\phi^2}, \tag{14}$$

we find the ratio becomes $\mathcal{R} \sim 1/c\kappa^2\phi^2$ when the above relation holds. Since the e-folding number is crudely given by $N \sim \kappa^2\phi^2$ and the scale observed through CMB corresponds to $N \sim \mathcal{O}(100)$, we have typically $\kappa\phi \sim \mathcal{O}(10)$. Hence, the ratio goes $\mathcal{R} \sim 10^{-2}$. Thus we find that the effect of the vector filed in (12) is negligible even when it is comparable with that of the scalar field in (13).

It turns out that the above situation is not transient one but an attractor. Suppose that ρ_A is initially negligible, $\mathcal{R}_i \ll 10^{-2}$. In the first slow-roll inflationary phase (9), the relation $e^{-\kappa^2\phi^2} \propto e^{4\alpha}$ holds as was shown in (10). Hence, the ratio $\mathcal{R}$ varies as $\mathcal{R} \propto e^{4(c-1)\alpha}$. As we now consider $c > 1$, ρ_A increases rapidly during inflation and eventually reaches $\mathcal{R} \sim 10^{-2}$. Whereas, when $\mathcal{R}$ exceeds 10^{-2}, the inflaton climbs up the potential due to the effect of the vector field in (13); hence ρ_A will decrease rapidly and go back to the value $\mathcal{R} \sim 10^{-2}$. Thus irrespective of initial conditions, ρ_A will track ρ_ϕ.

The above arguments tell us that the inflaton dynamics after tracking is governed by the modified slow-roll equations

$$\dot{\alpha}^2 = \frac{\kappa^2}{6}m^2\phi^2, \tag{15}$$

$$3\dot{\alpha}\,\dot{\phi} = -m^2\phi + c\kappa^2\phi p_A^2 e^{-c\kappa^2\phi^2-4\alpha}. \tag{16}$$

We refer to the phase governed by the above equations as the second inflationary phase, compared to the first one governed by the Eqs. (9). Using above equations, we can deduce

$$\phi\frac{d\phi}{d\alpha} = -\frac{2}{\kappa^2} + \frac{2cp_A^2}{m^2}e^{-c\kappa^2\phi^2-4\alpha}. \tag{17}$$

This can be integrated as $e^{-c\kappa^2\phi^2-4\alpha} = m^2(c-1)/c^2\kappa^2 p_A^2[1 + De^{-4(c-1)\alpha}]^{-1}$, where D is a constant of integration. This solution rapidly converges to

$$e^{-c\kappa^2\phi^2-4\alpha} = \frac{m^2(c-1)}{c^2\kappa^2 p_A^2}. \tag{18}$$

Thus, we found ρ_A becomes constant during the second inflationary phase. Substituting the result (18) into the modified slow-roll equation (16), we obtain the equation for the second inflationary phase

$$3\dot{\alpha}\,\dot{\phi} = -\frac{m^2}{c}\phi. \tag{19}$$

This indicates that $\dot{\phi}$ in the second phase of inflation is about $1/c$ times that in the first phase of inflation. In Fig. 1, we can see the value of $\dot{\phi}$ after the phase transition is about a half of that in the first phase, which agrees with the analytical estimate for $c = 2$.

Now let us consider the anisotropy. In the second slow-roll phase, Eq. (6) reads

$$3\dot{\alpha}\,\dot{\sigma} = \frac{\kappa^2 p_A^2}{3}e^{-c\kappa^2\phi^2-4\alpha}, \tag{20}$$

where we have assumed $\sigma \ll c\kappa^2\phi^2$, $\ddot{\sigma} \ll \dot{\alpha}\,\dot{\sigma}$. Using Eqs. (15) and (20), the anisotropy turns out to be determined by the ratio (14) as

$$\frac{\Sigma}{H} = \frac{\kappa^2 p_A^2 e^{-c\kappa^2\phi^2-4\alpha}}{9\dot{\alpha}^2} = \frac{2}{3}\mathcal{R}(t). \tag{21}$$

From Eq. (18), we can calculate the ratio

$$\mathcal{R}(t) = \frac{c-1}{c^2\kappa^2\phi^2}. \tag{22}$$

Using this relation, we can relate degrees of anisotropy to the slow-roll parameter as follows. Combining Eqs. (4) with (5), we obtain

$$\ddot{\alpha} = -\frac{\kappa^2}{2}\dot{\phi}^2 - \frac{\kappa^2}{3}e^{-c\kappa^2\phi^2-4\alpha}p_A^2, \tag{23}$$

where we have used $\dot{\sigma}^2 \ll \kappa^2\dot{\phi}^2$ derived from Eqs. (15), (19), (21), and (22). Thus, the slow-roll parameter is given by

$$\epsilon \equiv -\frac{\ddot{\alpha}}{\dot{\alpha}^2} = \frac{2}{c\kappa^2\phi^2}, \tag{24}$$

where we used the results (15), (18), and (19). Thus, combining Eqs. (21), (22), and (24), we reach a main result

$$\frac{\Sigma}{H} = \frac{1}{3}\frac{c-1}{c}\epsilon. \tag{25}$$

This remarkable relation shows a quite good agreement with the numerical results in Fig. 2.

Generality.—Although the discussion we have made so far is restricted to a specific form of potential V, we now argue that our finding is the general feature of the inflationary scenario in the presence of the vector field.

Let us consider the general potential $V(\phi)$ for the inflaton. Then, the coupling function should be of the form (11). Hence, in the slow-roll phase, the equation for the inflaton (7) becomes

$$3\dot{\alpha}\,\dot{\phi} = -V' + 2c\kappa^2\frac{V}{V'}f^{-2}p_A^2 e^{-4\alpha-4\sigma}. \tag{26}$$

When $c > 1$, the energy density of the vector will soon catch up with that of the inflaton. At the tracking point, ρ_A and ρ_ϕ tend to be $\rho_A \simeq (V'/V)^2\rho_\phi/4c\kappa^2$. Note that the slow-roll parameter now becomes

$$\epsilon \equiv -\frac{\ddot{\alpha}}{\dot{\alpha}^2} \simeq \frac{1}{2c\kappa^2}\left(\frac{V'}{V}\right)^2. \tag{27}$$

Then, again, we can conclude that the anisotropy becomes of the order of the slow-roll parameter:

$$\frac{\Sigma}{H} \simeq \frac{1}{6c\kappa^2}\left(\frac{V'}{V}\right)^2 \simeq \frac{1}{3}\epsilon. \tag{28}$$

Thus, we have shown that the anisotropy is universally determined by the slow-roll parameter. This is reminiscent of non-Gaussianity in single inflaton models [12].

Conclusion.—We have proposed an inflationary scenario with anisotropy. Remarkably, we have found that degrees of anisotropy are universally determined by the slow-roll parameter of inflation. Since the slow-roll parameter is observationally known to be of the order of a percent, the anisotropy during inflation cannot be entirely negligible. Indeed, we can expect rich phenomenology as consequences of the anisotropy during inflation. First of all, since the rotational invariance is violated, the statistical anisotropy of CMB temperature fluctuations can be expected [13]. More interestingly, tensor perturbations could be induced from curvature perturbations through the anisotropy of the background spacetime. One immediate consequence is a correlation between curvature and tensor perturbations [6]. This correlation should be detected through the analysis of temperature-B-mode correlation in CMB. Moreover, because of the anisotropy, there might be linear polarization in primordial gravitational waves. This polarization can be detected either through CMB observations or direct interferometer observations. These predictions can be checked by future observations. Theoretically, we need more systematic checks such as quantum loop effects [14].

Finally, let us point out another view of our result. Our finding of hairy inflation can be regarded as a counterexample to the cosmic no-hair conjecture. This hair stems from the fact that the inflation is not exactly de Sitter expansion. In fact, degrees of anisotropy are determined by the slow-roll parameter. In a sense, this is the origin of the universality of a percent level of vector hair.

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