Time-Reversal-Invariant Topological Superconductors and Superfluids in Two and Three Dimensions

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We construct time-reversal invariant topological superconductors and superfluids in two and three dimensions. These states have a full pairing gap in the bulk, gapless counterpropagating Majorana states at the boundary, and a pair of Majorana zero modes associated with each vortex. The superfluid ^{3}He B phase provides a physical realization of the topological superfluidity, with experimentally measurable surface states protected by the time-reversal symmetry. We show that the time-reversal symmetry naturally emerges as a supersymmetry, which changes the parity of the fermion number associated with each time-reversal invariant vortex and connects each vortex with its superpartner.

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The quantum Hall (QH) effect provides the first example of a topologically nontrivial state of matter, where the quantized Hall conductance is a topological invariant [1]. Analogously, chiral superconductors in a time-reversal symmetry breaking (TRB) $(p_x + ip_y)$ pairing state in $2d$ have a sharp topological distinction between the strong and weak pairing regimes [2]. In the weak pairing phase, the system has a full bulk gap and gapless chiral Majorana states at the edge, which are topologically protected. Chiral superconductors are analogous to the QH state—they both break time reversal (TR) and have chiral edge states. However, the edge states of a chiral superconductor have only half the degrees of freedom compared to the QH state, since the negative energy quasiparticle operators describe the same excitations as the positive energy ones. Moreover, a Majorana zero mode is trapped in each vortex core [2], which leads to a ground state degeneracy of 2^{n-1} as well as non-Abelian statistics in the presence of $2n$ vortices [3].

Recently, the quantum spin Hall (QSH) state [4,5] has been theoretically predicted [6] and experimentally observed in HgTe quantum wells [7]. As a time-reversal invariant (TRI) generalization of the QH state, the QSH state is characterized by a Z_2 topological number [8] and gapless helical edge states, where time-reversed partners counterpropagate [9,10]. Given the analogy between the chiral superconducting state and the QH state, and with the recent discovery of the TRI QSH state, it is natural to generalize the chiral pairing state to the helical pairing superconducting state, where fermions with up spins are paired in the $(p_x + ip_y)$ state, and fermions with down spins are paired in the $(p_x - ip_y)$ state. Such a TRI state has a full gap in the bulk, and counterpropagating helical Majorana states at the edge (in contrast, the edge states of the TRI topological insulator are helical *Dirac* fermions with twice the degrees of freedom). As is the case for the QSH state, a mass term for the edge states is forbidden by TR symmetry, and therefore, such a superconducting phase is topologically protected in the presence of time-reversal symmetry, and can be described by topological quantum numbers. (See also Refs. [11,12]). The four types of $2d$ topological states of matter discussed here are summarized in Fig. 1. Following the generalization of the QSH state to $3d$ [13–15], the $3d$ generalization of the TRI topological superconducting state can be obtained straightforwardly, and has $2d$ gapless Majorana surface states protected by TR symmetry. The TRI superconductor can be classified by a Z_2 topological number in 2D, and an integer Z topological number in 3D. The later case differs from the classification of the 3D topological insulator, due to distinct properties of the Majorana fermions.

Recently, the QSH state has been shown to lead to spin-charge separation [16,17]. In this work, we show that the $2d$ and $3d$ TRI topological superconductors are also characterized by a profound topological phenomena—an *emer-*

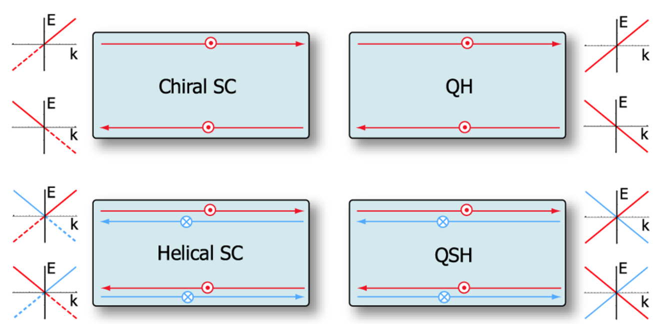

FIG. 1 (color online). (Top row) Schematic comparison of $2d$ chiral superconductor and the QH state. In both systems, TR symmetry is broken and the edge states carry a definite chirality. (Bottom row) Schematic comparison of $2d$ TRI topological superconductor and the QSH insulator. Both systems preserve TR symmetry and have a helical pair of edge states, where opposite spin states counterpropagate. The dashed lines show that the edge states of the superconductors are Majorana fermions so that the $E < 0$ part of the quasiparticle spectra are redundant.

gent supersymmetry. In $2d$, we show that a TRI topological defect of a Z_2 nontrivial superconductor carries a Kramers' pair of Majorana fermions. We prove the remarkable fact that in the presence of such a defect, the TR operator $\mathcal{T}$ changes the fermion number parity (or Witten index [18]), $\mathcal{T}^{-1}(-1)^{N_F}\mathcal{T} = -(-1)^{N_F}$ locally around the defect in the Z_2 nontrivial state, while preserving it, $\mathcal{T}^{-1}(-1)^{N_F}\mathcal{T} = (-1)^{N_F}$, in the Z_2 trivial state. Thus, given a TRI vortex, its time-reversed partner is actually its superpartner. A supersymmetric operation can be defined as an operation which changes the fermion number parity; therefore, in this precise sense, we show that the TR symmetry emerges as a supersymmetry in topological superconductors. This fact gives a precise physical definition of the Z_2 topological classification of any TRI superconductor and is generally valid in the presence of interactions and disorder. Our proposal offers the opportunity to experimentally observe supersymmetry in condensed matter systems without any fine tuning of microscopic parameters. In fact, one strong candidate for a realistic system is the BW phase of ^{3}He [19].

As the starting point, we consider a TRI p-wave superconductor with spin triplet pairing, which has the following 4×4 Bogoliubov–de Gennes Hamiltonian,

$$H = \frac{1}{2}\int d^2x \Psi^\dagger(x)\begin{pmatrix} \epsilon_\mathbf{p} I & i\sigma_2\sigma_\alpha\Delta^{\alpha j}p_j \\ \text{H.c.} & -\epsilon_\mathbf{p} I \end{pmatrix}\Psi(x) \quad (1)$$

with $\Psi(x) = (c_\uparrow(x), c_\downarrow(x), c_\uparrow^\dagger(x), c_\downarrow^\dagger(x))^T$, $\epsilon_\mathbf{p} = \mathbf{p}^2/2m - \mu$ the kinetic energy and chemical potential terms, and $\text{H.c.} \equiv (i\sigma_2\sigma_\alpha\Delta^{\alpha j}p_j)^\dagger$. The TR transformation is defined as $c_\uparrow \to c_\downarrow$, $c_\downarrow \to -c_\uparrow$. It can be shown that the Hamiltonian (1) is time-reversal invariant if $\Delta_{\alpha j}$ is a real matrix. To show the existence of a topological state, consider the TRI mean-field ansatz $\Delta^{\alpha 1} = \Delta(1, 0, 0)$, $\Delta^{\alpha 2} = \Delta(0, 1, 0)$. For such an ansatz, the Hamiltonian (1) is block diagonal with only equal spin pairing,

$$H = \frac{1}{2}\int d^2x \tilde{\Psi}^\dagger \begin{pmatrix} \epsilon_\mathbf{p} & \Delta p_+ & & \\ \Delta p_- & -\epsilon_\mathbf{p} & & \\ & & \epsilon_\mathbf{p} & -\Delta p_- \\ & & -\Delta p_+ & -\epsilon_\mathbf{p} \end{pmatrix}\tilde{\Psi} \quad (2)$$

with $\tilde{\Psi}(x) \equiv [c_\uparrow(x), c_\uparrow^\dagger(x), c_\downarrow(x), c_\downarrow^\dagger(x)]^T$, and $p_\pm = p_x \pm ip_y$. From this, we see that spin-up (down) electrons form $p_x + ip_y$ ($p_x - ip_y$) Cooper pairs, respectively.

In the weak pairing phase with $\mu > 0$, the $(p_x + ip_y)$ *chiral* superconductor is known to have chiral Majorana edge states propagating on each boundary, described by the Hamiltonian $H_{\text{edge}} = \sum_{k_y \geq 0} v_F k_y \psi_{-k_y}\psi_{k_y}$, where $\psi_{-k_y} = \psi_{k_y}^\dagger$ is the quasiparticle creation operator [2] and the boundary is taken parallel to the y direction. Thus, in analogy with the QSH system, we know that the edge states of the TRI system described by Hamiltonian (2) consist of spin-up and spin-down quasiparticles with opposite chirality:

$$H_{\text{edge}} = \sum_{k_y \geq 0} v_F k_y (\psi_{-k_y\uparrow}\psi_{k_y\uparrow} - \psi_{-k_y\downarrow}\psi_{k_y\downarrow}). \quad (3)$$

The quasiparticle operators $\psi_{k_y\uparrow}$, $\psi_{k_y\downarrow}$ can be expressed in terms of the eigenstates of the BdG Hamiltonian as $\psi_{k_y\uparrow} = \int d^2x[u_{k_y}(x)c_\uparrow(x) + v_{k_y}(x)c_\uparrow^\dagger(x)]$, $\psi_{k_y\downarrow} = \int d^2x[u_{-k_y}^*(x)c_\downarrow(x) + v_{-k_y}^*(x)c_\downarrow^\dagger(x)]$ from which the time-reversal transformation of the quasiparticle operators can be determined to be $\mathcal{T}^{-1}\psi_{k_y\uparrow}\mathcal{T} = \psi_{-k_y\downarrow}$, $\mathcal{T}^{-1}\psi_{k_y\downarrow}\mathcal{T} = -\psi_{-k_y\uparrow}$. In other words, $(\psi_{k_y\uparrow}, \psi_{-k_y\downarrow})$ transforms as a Kramers' doublet, which forbids a gap in the edge state spectrum when TR is preserved by preventing the mixing of the spin-up and spin-down modes. To see this explicitly, notice that the only k_y-independent term that can be added to the edge Hamiltonian (3) is $im\sum_{k_y}\psi_{-k_y\uparrow}\psi_{k_y\downarrow}$ with $m \in \mathcal{R}$. However, such a term is odd under TR, which implies that any backscattering between the quasiparticles is forbidden by TR symmetry. The discussion above is exactly parallel to the Z_2 topological characterization of the QSH system. In fact, the Hamiltonian (2) has exactly the same form as the four-band effective Hamiltonian proposed in Ref. [6] to describe HgTe quantum wells with the QSH effect. The edge states of the QSH insulators consist of an odd number of Kramers' pairs, which remain gapless under any small TR-invariant perturbation [9,10]. Such a "helical liquid" with an odd number of Kramers' pairs at the Fermi energy cannot be realized in any bulk $1d$ system, and can only appear as an edge theory of a $2d$ QSH insulator [9]. Similarly, the edge state theory Eq. (3) can be called a "helical Majorana liquid," and can only exist on the boundary of a Z_2 topological superconductor. Once such a topological phase is established, it is robust under any TRI perturbations such as Rashba-type spin-orbital coupling $H_R = \int d^2x c^\dagger(x)\lambda(\sigma_x p_y - \sigma_y p_x)c(x)$ and s-wave pairing $H_s = \int d^2x[\Delta_s c_\uparrow^\dagger c_\downarrow^\dagger(x) + \text{H.c.}]$, even if spin conservation is broken.

The Hamiltonian (1) can be easily generalized to three dimensions, in which case $\Delta^{\alpha j}$ becomes a 3×3 matrix with $\alpha = 1, 2, 3$ and $j = x, y, z$. An example of such a Hamiltonian is given by the well-known ^{3}He BW phase, for which the order parameter $\Delta^{\alpha j}$ is determined by an orthogonal matrix $\Delta^{\alpha j} = \Delta u^{\alpha j}$, $u \in \text{SO}(3)$ [19]. Here and below, we ignore the dipole-dipole interaction term [20] since it does not affect any essential topological properties. By applying a spin rotation, $\Delta^{\alpha j}$ can be diagonalized to $\Delta^{\alpha j} = \Delta\delta^{\alpha j}$, in which case the Hamiltonian (1) has the same form as a $3d$ Dirac Hamiltonian with momentum dependent mass $\epsilon(\mathbf{p}) = \mathbf{p}^2/2m - \mu$. We know that a band insulator described by the Dirac Hamiltonian is a $3d$ Z_2 topological *insulator* for $\mu > 0$ [13–15], and has nontrivial surface states. The corresponding superconductor Hamiltonian describes a topological *superconductor* with $2d$ gapless Majorana surface states. The surface theory can be written as

$$H_{\mathrm{surf}} = \frac{1}{2}\sum_{\mathbf{k}} v_F \psi^T_{-\mathbf{k}}(\sigma_z k_x + \sigma_x k_y)\psi_{\mathbf{k}} \qquad (4)$$

which remains gapless under any small TRI perturbation since the only available mass term $m\sum_{\mathbf{k}} \psi^T_{-\mathbf{k}}\sigma_y\psi_{\mathbf{k}}$ is time-reversal odd. Interestingly, surface Andreev bound states in the ^{3}He-BW phase have been observed experimentally , and if they have a topological origin, their properties will be sensitive to TR breaking like their counterparts, the QSH edge states [7]. The gapless nature of the surface states is only protected by the presence of TR symmetry, so applying a magnetic field should open a measurable gap in the surface state spectrum. Different from the $2d$ case, the $3d$ TRI topological superconductor is characterized by an integer [12] rather than Z_2, since a surface state theory with even number of surface Majorana cones can also be topologically stable. However, for the purpose of the present Letter, only the Z_2 part (even or odd number of surface Majorana cones) is important.

To illustrate the physical consequences of the nontrivial topology, we study the TRI topological defects of the topological superconductors. We start by considering the BdG Hamiltonian (2) in which spin-up and down electrons form $p_x + ip_y$ and $p_x - ip_y$ Cooper pairs, respectively. A TRI topological defect can be defined as a vortex of spin-up superfluid coexisting with an antivortex of spin-down superfluid at the same position. In the generic Hamiltonian (1), such a vortex configuration is written as $\Delta^{\alpha j} = \{\exp[i\sigma_2\theta(\mathbf{r}-\mathbf{r}_0)]\}^{\alpha j}$, $\alpha = 1,\ 2$, and $\Delta^{3j} = 0$, where $\theta(\mathbf{r}-\mathbf{r}_0)$ is the angle of $\mathbf{r}$ with respect to the vortex position $\mathbf{r}_0$. Since in the vortex core of a weak pairing $p_x + ip_y$ superconductor there is a single Majorana zero mode [2,22], one immediately knows that a pair of Majorana zero modes exist in the TRI vortex core. In terms of the electron operators, the two Majorana fermion operators can be written as $\gamma_\uparrow = \int d^2x[u_0(x)c_\uparrow(x) + u_0^*(x)c_\uparrow^\dagger(x)]$, $\gamma_\downarrow = \int d^2x[u_0^*(x)c_\downarrow(x) + u_0(x)c_\downarrow^\dagger(x)]$ where we have used the fact that the spin-down operator can be obtained from the time-reversal transformation of the spin-up one. The Majorana operators satisfy the anticommutation relation $\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}$, and the TR transformation of the Majorana fermions is $\mathcal{T}^{-1}\gamma_\uparrow\mathcal{T} = \gamma_\downarrow$, $\mathcal{T}^{-1}\gamma_\downarrow\mathcal{T} = -\gamma_\uparrow$. Similar to the case of the edge states studied earlier, the Majorana zero modes are robust under any small TRI perturbation, since the only possible term $im\gamma_\uparrow\gamma_\downarrow$ which can lift the zero modes to finite energy is TR odd, i.e., $\mathcal{T}^{-1}i\gamma_\uparrow\gamma_\downarrow\mathcal{T} = -i\gamma_\uparrow\gamma_\downarrow$.

From the two Majorana zero modes $\gamma_\uparrow$, $\gamma_\downarrow$, a complex fermion operator can be defined as $a = (\gamma_\uparrow + i\gamma_\downarrow)/2$, which satisfies the fermion anticommutation relation $\{a, a^\dagger\} = 1$. Since $\gamma_\uparrow$, $\gamma_\downarrow$ are zero modes, we obtain $[a, H] = 0$; i.e., a is the annihilation operator of a zero-energy quasiparticle. Thus, the ground state of the system is at least twofold degenerate, with two states $|G_0\rangle$ and $|G_1\rangle = a^\dagger|G_0\rangle$ containing 0 and 1 a-fermions. Since $a^\dagger a = (1 + i\gamma_\uparrow\gamma_\downarrow)/2$, the states $|G_{0(1)}\rangle$ are eigenstates of $i\gamma_\uparrow\gamma_\downarrow$ with eigenvalues $-1(+1)$, respectively. Thus, from the oddness of $i\gamma_\uparrow\gamma_\downarrow$ under TR, we know that $|G_0\rangle$ and $|G_1\rangle$ are time-reversed partners. In the superconductor, fermion number parity $(-1)^{N_F}$ is conserved, and all the eigenstates of the Hamiltonian can be classified by their values of $(-1)^{N_F}$. If, say, $|G_0\rangle$ is a state with $(-1)^{N_F} = 1$, then $|G_1\rangle = a^\dagger|G_0\rangle$ must satisfy $(-1)^{N_F} = -1$. Since $|G_0\rangle$ and $|G_1\rangle$ are time-reversal partners, we know that in the Hilbert space of the zero-energy states, the TR transformation changes the fermion number parity:

$$\mathcal{T}^{-1}(-1)^{N_F}\mathcal{T} = -(-1)^{N_F}. \qquad (5)$$

Equation (5) is the central result of this Letter. Since a transformation changing fermion number by an odd number is a "supersymmetry," the TR symmetry emerges as a discrete supersymmetry for each TRI topological defect. A TRI vortex and its time-reversed partner are actually superpartners. This has a striking consequence. Typically, when excitations and their time-reversed partners fuse, they form a boson, e.g., spin-up and down electrons with opposite momenta form Cooper pair bosons or a conventional s-wave vortex and antivortex pair annihilating into a bosonic vacuum. Here, however, because of the supersymmetric nature, a vortex and its time-reversed partner annihilate to leave behind a fermion as illustrated in Fig. 2. This occurs because the time-reversal operator becomes a supercharge which carries fermionic quantum numbers. Similar analysis applies to the edge theory (3),

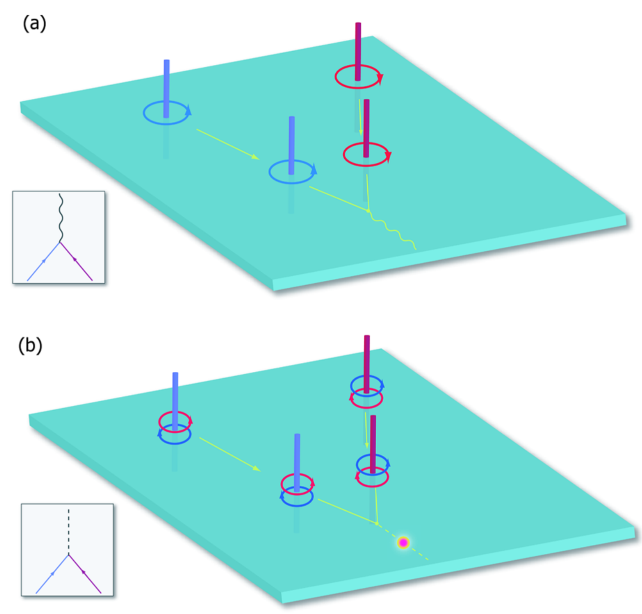

FIG. 2 (color online). (a) The fusion of two vortices with opposite vorticity in a conventional superconductor. The two vortices are time-reversal partners, and they fuse into a boson. (b) The fusion of two vortices with opposite vorticity in a Z_2 topological superconductor. The two vortices are superpartners, and they fuse into a fermion. The insets indicate schematic Feynman diagrams of these two processes.

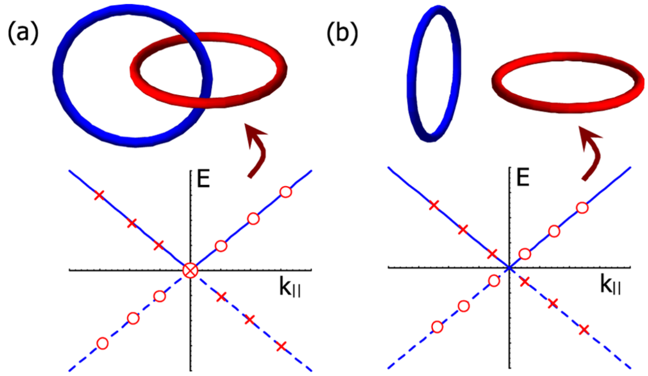

FIG. 3 (color online). Illustration of a $3d$ TRI topological superconductor with two TRI vortex rings which are (a) linked or (b) unlinked. The $E - k_{\|}$ dispersion relations show schematically the quasiparticle levels confined on the red vortex ring in both cases. "○" and "×" stand for the quasiparticle levels that are Kramers' partners of each other. Only case (a) has a pair of Majorana zero modes located on each vortex ring since the rings are linked.

which shows that in the $1d$ helical Majorana liquid is also a theory with TR symmetry as a discrete supersymmetry.

At a first glance, Eq. (5) seems to contradict the fundamental fact that the electron number of the whole system is invariant under TR. Such a paradox is resolved by noticing that there are always an even number of topological defects in a closed system without boundary. Under the TR transformation, the fermion number parity around each vortex core is odd, but the total fermion number parity remains even as expected. Equation (5) is a generic definition of TRI topological superconductors: A two-dimensional TRI superconductor is Z_2 nontrivial if and only if fermion number parity around a TRI topological defect is odd under TR.

All the conclusions above can be generalized to $3d$ topological superconductors and superfluids. In the ^{3}He BW phase, the Goldstone manifold of the order parameter is $\Delta^{\alpha j} = \Delta u^{\alpha j} \in SO(3) \times U(1)$ [19,23]. A time-reversal invariant configuration satisfies $\Delta^{\alpha j} \in \mathcal{R}$, which restricts the order parameter to $SO(3)$. TRI topological defects are $1d$ "vortex" rings and have a Z_2 classification [19]. By solving the BdG equations in the presence of such vortex rings, it can be shown that there are linearly dispersing quasiparticles propagating on each vortex ring. However, for a ring with finite length, the quasiparticle spectrum is discrete. Generically, there is no guarantee that a pair of Majorana modes exist at exactly zero energy. The existence of zero modes on the vortex rings turns out to be a topological property determined by the linking number between different vortex rings. Here we will write our conclusion and leave the details for a separate work: There are a robust pair of Majorana fermion zero modes confined on a vortex ring if and only if the ring is linked to an odd number of other vortex rings. Such a condition is shown in Fig. 3. Consequently, a physical definition of a TRI topological superconductor in $3d$ is: A $3d$ TRI superconductor is Z_2 nontrivial if and only if the fermion number parity around one of the two mutually linked TRI vortex rings is odd under TR.

In conclusion, we introduced the concepts of TRI topological superconductors and superfluids and showed that a supersymmetry emerges naturally in these systems as a consequence of the time-reversal symmetry. To be more precise, time-reversal symmetry relates two states (a Kramers pair) with the opposite fermion number. Since the time-reversal symmetry is ubiquitous, the resulting supersymmetry can be realized without fine tuning, independent of the microscopic details.

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