Relativistic Invariance of Lyapunov Exponents in Bounded and Unbounded Systems

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The study of chaos in relativistic systems has been hampered by the observer dependence of Lyapunov exponents (LEs) *and* of conditions, such as orbit boundedness, invoked in the interpretation of LEs as indicators of chaos. Here we establish a general framework that overcomes both difficulties and apply the resulting approach to address three fundamental questions: how LEs transform under Lorentz and Rindler transformations and under transformations to uniformly rotating frames. The answers to the first and third questions show that inertial and uniformly rotating observers agree on a characterization of chaos based on LEs. The second question, on the other hand, is an ill-posed problem due to the event horizons inherent to uniformly accelerated observers.

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The quest for an observer-independent characterization of chaos in relativistic systems [1] has been an intense area of research and promises to provide significant new insights into the properties of chaotic dynamics [2]. An important recent result [3] concerns the transformation of Lyapunov exponents (LEs) under spacetime diffeomorphisms. We recall that the dynamics of a bounded solution $\mathbf{X}(t)$ of a dynamical system

$$\frac{d\mathbf{X}}{dt} = \boldsymbol{F}(\mathbf{X}) \tag{1}$$

is chaotic if it presents sensitive dependence on initial conditions [4]. The associated LEs [5] are given by $\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \log\|\varphi_i(t)\|$, where $\varphi_i(t)$ are solutions of the linearized equation $\frac{d}{dt}\varphi_i = [D_{\mathbf{X}}\boldsymbol{F}(\mathbf{X}(t))]\varphi_i$. Positive LEs are related to exponential divergence of initially close trajectories and, consequently, to chaotic dynamics. For space diffeomorphisms $\mathbf{X} = \boldsymbol{\Psi}(\mathbf{Y})$, the invariance of the LEs is well established under rather general conditions (see, e.g., [6,7]). In contrast, for well-behaved *spacetime* diffeomorphisms involving time changes of the form $d\tau = \Lambda(\mathbf{X})dt$, it has been shown [3] that the LEs transform according to

$$\lambda_i^\tau = \lambda_i^t/\langle\Lambda\rangle_t, \tag{2}$$

where $0 < \langle\Lambda\rangle_t < \infty$ is the time average of Λ along the corresponding trajectory. Therefore, although the values of the LEs are themselves noninvariant, their signs are preserved and assure an invariant criterion for chaos under spacetime transformations. This result was obtained under conditions for which LEs are known to be valid quantifiers of chaos, of which the most limiting ones are the assumptions that the system has a natural invariant probability measure and the orbits are bounded both before and after the transformation.

In this Letter, we extend this result to an important class of transformations that do not preserve the boundedness of the orbits, to then address three long-outstanding questions. The first question is how the LEs transform under Lorentz transformations. This question determines whether all inertial observers agree on a LE-based characterization of chaos. We show that the answer is affirmative despite the fact that the dynamics becomes unbounded with respect to at least one of the reference frames. We use this example to establish an *extended boundedness* condition for the definition of the LEs as indicators of chaos, which is formulated relative to the trajectories themselves rather than a fixed point of the phase space. The second question is how the LEs behave under Rindler transformations, a question equivalent to asking whether uniformly accelerated observers agree on an inertial characterization of chaos based on LEs. We show that this question is ill posed because uniformly accelerated observers do not have access to the late-time dynamics. The latter relates to the fact that chaos and LEs are asymptotic concepts [8] whose definitions involve a limit $t \to \infty$. We also consider transformations to uniformly rotating frames, and show that the positivity of the LEs remains invariant under such transformations.

Our main result stems from this analysis and can be stated for any system and any spacetime diffeomorphic transformation, as follows. For the system written in autonomous form, the LEs transform according to Eq. (2) and remain invariant indicators of chaos if, as shown below, (i) our extended boundedness condition is satisfied, (ii) the Jacobian of the transformation is bounded, and (iii) Λ is positive for all t and $0 < \langle\Lambda\rangle_t < \infty$. These conditions depend not only on the transformation properties of the dynamical variables $\mathbf{X}$ and the change of reference frames but also on the choice of spacetime coordinates. They are satisfied for global nonsingular transformations of bounded orbits for which $\inf\Lambda^{\pm 1} > 0$, whether the system is conservative or dissipative, mechanical or not. These conditions clarify previous results [9] that seem to challenge the invariance of chaos for relativistic observers, and show that LEs lead to invariant conclusions about chaos.

We first note that under a space diffeomorphism $\mathbf{X} = \mathbf{\Psi}(\mathbf{Y})$, system (1) is mapped into $\frac{d}{dt}\mathbf{Y} = [D_{\mathbf{Y}}\mathbf{\Psi}(\mathbf{Y})]^{-1} \times \boldsymbol{F}(\mathbf{\Psi}(\mathbf{Y}))$, rendering the solutions of the new linearized dynamics to be related to those of (1) as $\varphi_i(t) = [D_{\mathbf{Y}}\mathbf{\Psi}(\mathbf{Y}(t))]\tilde{\varphi}_i(t)$ [7]. Hence, the corresponding LEs satisfy

$$\liminf_{t\to\infty}\frac{1}{t}\log\frac{\|[D_{\mathbf{Y}}\mathbf{\Psi}(\mathbf{Y}(t))]\tilde{\varphi}_i(t)\|}{\|\tilde{\varphi}_i(t)\|} \leq \lambda_i - \tilde{\lambda}_i \leq \limsup_{t\to\infty}\frac{1}{t}\log\frac{\|[D_{\mathbf{Y}}\mathbf{\Psi}(\mathbf{Y}(t))]\tilde{\varphi}_i(t)\|}{\|\tilde{\varphi}_i(t)\|}. \quad (3)$$

Suppose the solutions $\mathbf{X}(t)$ are limited to a compact subset of the space. Since the diffeomorphism maps bounded solutions $\mathbf{X}(t)$ into bounded solutions $\mathbf{Y}(t)$, matrix $D_{\mathbf{Y}}\mathbf{\Psi}(\mathbf{Y}(t))$ is nonsingular and, besides, there are finite nonzero constants $L^{\pm} = \sup\|[D_{\mathbf{Y}}\mathbf{\Psi}(\mathbf{Y}(t))]^{\pm 1}\|$ leading to

$$\lim_{t\to\infty}\frac{1}{t}\log\frac{1}{L^-} \leq \lambda_i - \tilde{\lambda}_i \leq \lim_{t\to\infty}\frac{1}{t}\log L^+, \quad (4)$$

which imply $\tilde{\lambda}_i = \lambda_i$ [7]. This argument explores the boundedness of $\mathbf{X}(t)$ and $\mathbf{Y}(t)$. Below we study (4) and the existence of $L^{\pm}$ when this condition is relaxed.

We consider transformations of reference frame in which (1) describes a bounded autonomous system with respect to the initial (inertial) observers. More general transformations can be obtained by a composition of such transformations. We start with single-particle systems. While general relativity allows arbitrary spacetime coordinates, and conditions (i)–(iii) can be applied to any of them, we will assume that the dynamics is described in terms of physical times (i.e., the time measured by observers at rest in the reference frame).

Lorentz transformations.—We first focus on the case in which function $\boldsymbol{F}$ depends only on the configuration-space coordinates, such as in the evolution of a fluid element determined by a stream function, and consider a Lorentz boost with velocity v along the x direction, $(ct, x, y, z) \to (ct', x', y', z') = \mathbf{\Psi}^{-1}(ct, x, y, z)$, where

$$\mathbf{\Psi}^{-1}(ct, x, y, z) = (\gamma(ct - vx/c), \gamma(x - vt), y, z) \quad (5)$$

for $\gamma = 1/\sqrt{1-(v/c)^2}$. We focus on the space spanned by $(ct, x, y, z) \equiv (ct, \boldsymbol{x})$, where we have enlarged the configuration space in order to incorporate ct as a new coordinate. The extended version of (1) then reads

$$\frac{d}{dt}\begin{pmatrix} w \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} c \\ \boldsymbol{F}(\boldsymbol{x}) \end{pmatrix}, \quad (6)$$

where $dw/dt \equiv d(ct)/dt$.

The main advantage of this formulation is that the transformed system remains autonomous and the spacetime transformation can be reduced to an ordinary space diffeomorphism; it can be split as $\boldsymbol{T} \circ \boldsymbol{S}(ct, \boldsymbol{x})$, where $\boldsymbol{S}$ is a transformation $(w', \boldsymbol{x}') = \mathbf{\Psi}^{-1}(w, \boldsymbol{x})$ that preserves the independent variable and $\boldsymbol{T}$ is a time redefinition $dt' = \Lambda(w, \boldsymbol{x})dt$. (Another advantage is that the analysis extends immediately to $\boldsymbol{F}$ with explicit time-periodic dependence.) The solutions of (6) are unbounded along the w direction, but this is not a problem since the nonzero LEs of system (6) are identical to those of (1).

There is a caveat, however: the *spatial* boundedness of the solutions is not preserved under Lorentz transformations. A trajectory confined to a bounded spacelike region ($\sup\|\boldsymbol{x}(t)\| < \infty$) of the first reference frame is seen as spatially unbounded from the other inertial reference frame. Similar problem is observed even for Galilean transformations, but in classical dynamics one can adopt a reference frame where the solutions are bounded. In relativistic dynamics such a choice would raise questions about the invariance of the LEs, which is precisely the object of this Letter.

To proceed we first make the crucial observation that the study of chaos *can* be extended to this class of spatially unbounded orbits, even though the same does not hold true for unbounded systems in general. Indeed, sensitive dependence on initial conditions and LEs depend exclusively on the relative time evolution between nearby trajectories; their dependence on the reference frame is limited to the definition of the spacetime coordinates used to measure the distances between the neighboring trajectories as they evolve over identical time intervals. Therefore, chaos can be properly defined and LEs can be used as indicators of chaos along an unbounded trajectory $\boldsymbol{y}(t)$ insofar as $\|\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)\|$ remains uniformly upper bounded for all t and all trajectories $\hat{\boldsymbol{y}}(t)$ with initial conditions in a neighborhood of $\boldsymbol{y}(0)$. That is, our condition is that the evolution of a small ball of points will remain bounded with respect to the local observers at position $\boldsymbol{y}(t)$, regardless of whether it remains bounded with respect to a fixed point of the reference frame. We refer to this as the extended boundedness condition. This condition is satisfied for $\boldsymbol{y}(t)$ interpreted as the extended coordinates $(w'(t), \boldsymbol{x}'(t))$ after the transformation $\boldsymbol{S}$ whenever the original system (1) is bounded.

Having shown that LEs remain valid indicators of chaos, we now turn to the effect of the Lorentz transformations on the LEs. For the transformation $\boldsymbol{T}$, from Eq. (5) we have

$$dt' = \gamma\left(1 - \frac{v}{c^2}F_x(\boldsymbol{x}(t))\right)dt \equiv \Lambda(\boldsymbol{x}(t))dt, \quad (7)$$

where $F_x(\boldsymbol{x})$ stands for the x component of $\boldsymbol{F}(\boldsymbol{x})$. For $|F_x(\boldsymbol{x}(t))| \leq c$, implying $\inf\Lambda(\boldsymbol{x}(t)) > 0$ in the present case, we have $0 < \langle\Lambda\rangle_t = \lim_{t\to\infty}\frac{t'(t)}{t} = \lim_{t\to\infty}\frac{1}{t} \times \int_0^t \Lambda(\boldsymbol{x}(p))dp < \infty$ [10]. This allows us to factor the LEs transformed by $\boldsymbol{T} \circ \boldsymbol{S}$ as

$$\tilde{\lambda}_i^{t'} = \tilde{\lambda}_i^t/\langle\Lambda\rangle_t, \quad (8)$$

where $\langle\Lambda\rangle_t$ is the contribution due to $\boldsymbol{T}$ and $\tilde{\lambda}_i^t$ corresponds to λ_i^t transformed by $\boldsymbol{S}$. The problem is thus reduced to the transformation of the LEs under the spatial transformation $\boldsymbol{S}$. The nonsingular nature of (5) assures the existence of the constants $L^{\pm}$ necessary to establish the bounds in (4) because, irrespective of the spatial unboundedness, the Jacobian $D_y\Psi(y)$ of the transformation is bounded. Applying the Euclidean norm to $D_y\Psi(y)$, we obtain $L^+ = L^- = \sqrt{(c+|v|)/(c-|v|)}$, leading to $\tilde{\lambda}_i^t = \lambda_i^t$. In particular, all positive LEs remain positive under this transformation.

Combined with Eq. (8), this results in $\tilde{\lambda}_i^{t'} = \lambda_i^t/\langle\Lambda\rangle_t$, which is precisely the transformation (2) previously established for bounded orbits [3]. A different result was presented in [9] for averages over local LEs [11], but that is because that study was restricted to time dilatations and length contractions, which correspond to the transformation of a dynamical variable such as volume for the time measured at a fixed point of the reference frame, whereas our analysis describes single-particle dynamics for the time measured at the position of the particle.

If system (1) involves the evolution of velocities, as expected for a particle in a 3D potential, the Lorentz transformation (5) must be extended to include the transformation of $\boldsymbol{u} \equiv d\boldsymbol{x}/dt$ into $\boldsymbol{u}' \equiv d\boldsymbol{x}'/dt'$, which is given by $u_x' = \eta(u_x - v)$, $u_y' = \eta\gamma^{-1}u_y$, and $u_z' = \eta\gamma^{-1}u_z$, where $\eta = 1/(1 - u_x v/c^2)$. The transformation $(w, \boldsymbol{x}, \boldsymbol{u}) \rightarrow (w', \boldsymbol{x}', \boldsymbol{u}')$ satisfies the extended boundedness condition and has constants $0 < L^{\pm} < \infty$, as long as $|v| < c$ and $|u_x(t)| \leq c$. This ensures that the LEs will be transformed as in (2).

Rindler transformations.—With respect to an inertial reference frame, an observer with constant proper acceleration a along the x direction has a hyperbolic worldline given by $ct(\tau) = \frac{c^2}{a}\sinh\frac{a\tau}{c}$, $x(\tau) = \frac{c^2}{a}\cosh\frac{a\tau}{c}$, where τ stands for the observer's proper time. The corresponding Rindler transformation [12] is defined by $(ct, x, y, z) \rightarrow (c\tau(t, x), \xi(t, x), y, z)$, with

$$ct(\tau, \xi) = c\sqrt{\frac{2\xi}{a}}\sinh\frac{a\tau}{c}, \quad x(\tau, \xi) = c\sqrt{\frac{2\xi}{a}}\cosh\frac{a\tau}{c}, \qquad (9)$$

and $\xi > 0$ (see Fig. 1). In contrast with the Lorentz case, the Rindler transformations are nonlinear in x and ct.

Focusing on the space defined by the extended configuration-space coordinates, matrix $D_y\Psi(y)$ and its inverse for the Rindler transformation of (6) have unit determinants but their largest eigenvalues diverge as $c/\sqrt{2a\xi}\cosh a\tau/c$ for $\xi \rightarrow 0$. Therefore, one cannot identify finite constants $L^{\pm}$ that could be used to compare $\tilde{\lambda}_i^t$ and λ_i^t. This behavior can be interpreted in terms of our extended boundedness condition, which is not satisfied in this case because $(c\tau(t, x(t)), \xi(t, x(t)), y(t), z(t))$ diverges at the light cone and is undefined beyond it. Moreover, from the inverse of Eq. (9), we have

$$d\tau = \frac{c^2}{a}\left[\frac{x(t) - tF_x(\boldsymbol{x}(t))}{x(t)^2 - (ct)^2}\right]dt \equiv \Lambda(ct, \boldsymbol{x}(t))dt, \qquad (10)$$

where $\Lambda(ct, \boldsymbol{x}(t))$ diverges when the original solution $(ct, \boldsymbol{x}(t))$ crosses the light cone $x^2 = c^2t^2$. (The same holds true for the physical time, $dt' = \sqrt{2a\xi/c^2}d\tau$.) The average

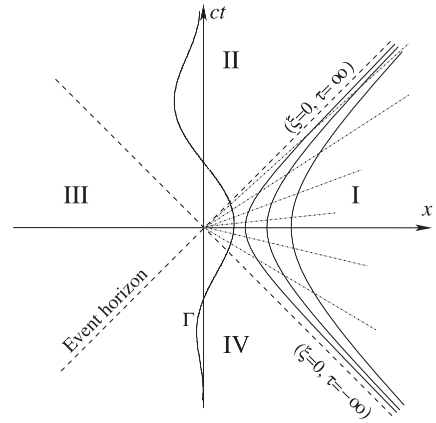

FIG. 1. Accessibility to the dynamics is observer dependent. The uniformly accelerated observers are unaware of all events occurring in regions II and III of the original Minkowski spacetime. They only access the dynamics of a trajectory Γ during the time the trajectory crosses region I [13]. If the trajectory is spatially bounded with respect to the original observers, as assumed for system (1), this corresponds to an infinite time interval $\Delta\tau$ but only to a finite time interval Δt. The hyperboles and dotted lines correspond, respectively, to constant ξ and constant τ.

$\langle\Lambda\rangle_t$ is not well defined and, as a result, the Rindler transformed system does not have a natural probability measure against which the LEs could be calculated [3]. Therefore, the question of how the LEs transform under Rindler transformations is ill posed.

The origin of the problem is the horizon structure (and its counterpart for $t \rightarrow -t$) inherent to uniformly accelerated observers [12]. The transformation (9) is not a global spacetime diffeomorphism since it maps only one-quarter of the Minkowski spacetime, as shown in Fig. 1. Any event located in regions II and III will never reach the accelerated observers. While singularities can be an artifact of the coordinates, event horizons are an attribute of the reference frame. They prevent the Rindler observers from having access to the asymptotic dynamics of the original system and hence from formulating a criterion for chaos—based on the observation of individual trajectories—that is valid for the inertial observers [13]. It is interesting that such a problem, related to the global structure of the spacetime, manifests itself as a violation of our conditions for the transformation of LEs.

Because the late-time dynamics of the dilated time $\tau \rightarrow \infty$ does not correspond to the late-time dynamics of the original system, even if one could compute the LEs as seen from the accelerated frame [9], this would be a problem different from the originally proposed one. This is analogous to the limits imposed by the cosmological singularity to the determination of chaos in Freidmann-Robertson-Walker cosmologies [8] and is also predicted for Rindler transformations of any other dynamical system and for any choice of coordinates.

Rotating frames.—The crucial role played by the event horizon in the Rindler case can be better appreciated if one considers a physical situation involving a nonlinear transformation that does not introduce event horizons. This is precisely the case of uniformly rotating frames [14]: $r' = r$, $\theta' = \theta + \Omega t$, $z' = z$, and $cdt' = [g(r) + \Omega^2 r^2/g(r)]dt + [\Omega r^2/g(r)]d\theta$, where $g(r) = \sqrt{c^2 - \Omega^2 r^2}$, Ω is a constant, and t' is the physical time in the rotating frame [15]. This leads to

$$dt' = \left(\frac{g(r(t))}{c} + \frac{\Omega r^2(t)[\Omega + F_\theta(\boldsymbol{x}(t))]}{cg(r(t))}\right)dt, \tag{11}$$

where $F_\theta(\boldsymbol{x}) = d\theta/dt$. The transformation of the LEs of (6) is in this case well defined since the extended boundedness condition is satisfied for orbits in closed sets of the physical region $|\Omega|r < c$ for which $-\Omega r^2 F_\theta(\boldsymbol{x}) < c^2$, where both the function $\Lambda(\boldsymbol{x})$ and the constants $L^\pm$ are upper and lower bounded away from zero. The latter follows from the fact that the entries of the Jacobian matrix $D_{\boldsymbol{y}}\boldsymbol{\Psi}(\boldsymbol{y})$ and its inverse for the transformation $(ct, r, \theta, z) \to (ct', r', \theta', z')$ are all continuous for $\Omega r < c$. A subtlety here is that in rotating frames the differential dt' of the physical time is not exact and cannot be integrated globally, meaning that the Jacobian elements involving derivatives of ct' must be determined from cdt' in the immediate neighborhood of a given r. The transformation $t \to t'$ is defined locally but it can always be extended along the corresponding trajectories. Therefore, the LEs transform as predicted by (2) also for rotating frames.

Generalization and discussion.—Our derivation of Eq. (8) also demonstrates that conditions (i)–(iii) are sufficient (and usually necessary) for the validity of (2) in general. Indeed, while we considered specific transformations and specific classes of dynamical systems in our explicit examples, these three conditions are precisely the conditions we have to verify for any system and any transformation. The extended boundedness condition—satisfied both before and after the transformation in the extended space, which includes ct as an additional coordinate—guarantees that the system can be kept autonomous and that LEs remain valid indicators of chaos. The condition that the Jacobian is bounded—in the sense of having positive finite constants $L^\pm$ for the transformation in the extended space—ensures the validity of the identity $\tilde{\lambda}_i^t = \lambda_i^t$. Finally, Λ and $\langle\Lambda\rangle_t$ positive and finite—again, in the extended space—guarantees that the time transformation is well defined and the signs of the LEs are conserved; it also guarantees that the time transformation is invertible, a condition we saw violated for the Rindler transformation.

These conditions are readily applicable to any system and any change of reference frame and coordinates. The latter includes the choice of the time parameter or of the observers in the reference frame with respect to which the time is measured. In the examples above, the dynamical system describes a single particle, the dynamical variables represent the coordinates and possibly velocities of the particle, and the time is assumed to be recorded locally—at each instant by the observer in the reference frame that is at the point where the particle is. However, other choices are equally valid. For a many-particle system under Lorentz transformation, for example, the time could be measured, e.g., with respect to the position of one of the particles, $dt' = \gamma[1 - \frac{v}{c^2}F_{x_i}(\boldsymbol{x}(t))]dt$, with respect to the center of mass, $dt' = \gamma[1 - \frac{v}{c^2}\sum_i(m_i/\sum_j m_j)F_{x_i}(\boldsymbol{x}(t))]dt$, or with respect to a fixed point, $dt' = \gamma dt$. Moreover, the dynamical system can describe a process whose dynamical variables do not necessarily correspond to coordinates and velocities in the physical space. In this general case the system can be written as $\frac{d}{dt}\mathrm{X}_i = F_i(\mathrm{X}_1, \ldots, \mathrm{X}_n)$, $i = 1, \ldots, n$, and the transformation is locally defined as $(cdt, d\mathrm{X}_1, \ldots, d\mathrm{X}_n) \to (cdt', d\mathrm{X}_1', \ldots, d\mathrm{X}_n')$. The latter is determined by the change of reference frame and space-time coordinates, $(cdt, d\boldsymbol{x}) \to (cdt', d\boldsymbol{x}')$, and depends on the nature of the dynamical variables, i.e., whether they transform as scalars, vectors, tensors, or in a different way. The choice of observers in the new reference frame is always accounted for through the choice of $\frac{d}{dt}\boldsymbol{x}$ in the transformation formula $dt' = [\frac{\partial}{\partial t}t'(ct, \boldsymbol{x}) + \nabla_{\boldsymbol{x}} t'(ct, \boldsymbol{x}) \cdot \frac{d}{dt}\boldsymbol{x}]dt$, where this term vanishes only if the time is measured (remotely) by a fixed observer.

The results are thus general and account for properties inherent to relativistic observers, such as event horizons and spatial unboundedness. In particular, they apply to both inertial and noninertial reference frames and do not involve the identification of privileged observers.

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