Study of the de Almeida–Thouless Line Using Power-Law Diluted One-Dimensional Ising Spin Glasses

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We test for the existence of a spin-glass phase transition, the de Almeida–Thouless line, in an externally applied (random) magnetic field by performing Monte Carlo simulations on a power-law diluted one-dimensional Ising spin glass for very large system sizes. We find that a de Almeida–Thouless line occurs only in the mean-field regime, which corresponds, for a short-range spin glass, to dimension d larger than 6.

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Perhaps the most surprising prediction of the mean-field theory of spin glasses is that an Ising spin glass has a line of transitions in an external magnetic field, the de Almeida–Thouless (AT) [1] line. This instability line separates a high-temperature high-field paramagnetic phase where relaxation times—possibly very large—stay finite from a low-temperature low-field phase where the energy landscape has valleys separated by truly infinite barriers in the thermodynamic limit. The AT line, an ergodic to nonergodic transition with no change in symmetry, is perhaps the most striking prediction of the mean-field theory of spin glasses. Whether or not it occurs in realistic systems is a major unsolved problem.

The existence or otherwise absence of an AT line in real (short-range) spin glasses is also a key feature distinguishing the two most popular scenarios for the nature of the spin-glass state below the (zero-field) transition temperature: the replica-symmetry breaking (RSB) picture of Parisi [2] and the "droplet picture" of Fisher and Huse [3,4]. The RSB picture assumes that the behavior of real spin glasses is very similar to that of the mean-field solution [2] of the Sherrington-Kirkpatrick infinite-range model. Since the mean-field model has a stable spin-glass state in a field and thus has an AT line, it is proposed that this also occurs for any short-range system with a finite-temperature transition in zero field. By contrast, the droplet picture makes certain assumptions about the nature of the low-energy, large-scale excitations (droplets) from which one finds no AT line in any dimension.

Experimentally, it has been harder to determine if an AT line occurs than to show that there is a transition in zero field. For the latter case, the divergence of the nonlinear susceptibility provides a clear signature of the transition. Unfortunately, the nonlinear susceptibility does not diverge in a field, i.e., along the AT line. However, as noted by two of us [5], there is a closely related static quantity which diverges on the AT line and which can be measured in simulations, albeit not in experiments. A finite-size scaling analysis of the two-point correlation length indicated the absence of an AT line for three-dimensional (3D) Ising spin glasses [5,6]. Subsequently, the same idea was applied to a one-dimensional (1D) model by Katzgraber and Young (KY) [7], in which every spin interacts with every other spin in the system with a strength which falls off with a power of the distance. By varying the power, one can simulate the whole range of possible behaviors [4,7,8], from infinite-range through mean-field to non-mean-field and finally to the absence of a finite-temperature transition. This is analogous to changing the space dimension d of short-range finite-dimensional models. KY found that an AT line does occur for parameter values corresponding to the mean-field case (for short-range systems that would be for $d \geq 6$) but not in the non-mean-field case ($d < 6$). The possibility of a critical dimension above which the AT line occurs had been considered before; see, for example, the discussion in Ref. [9].

Model and observables.—The model studied by KY is fully connected so the CPU time for one Monte Carlo sweep (MCS) grows as $\mathcal{O}(L^2)$, where L is the number of spins. This is inefficient for large L. Recently, this difficulty was removed in an elegant way in Ref. [10] by diluting the interactions and fixing the connectivity z. We thus study

$$\mathcal{H} = -\sum_{i,j} \varepsilon_{ij} J_{ij} S_i S_j - \sum_i h_i S_i, \qquad (1)$$

where $S_i = \pm 1$ are Ising spins evenly distributed on a ring of length L in order to ensure periodic boundary conditions. The sum is over all spins on the chain, and the couplings J_{ij} are normally distributed with zero mean and standard deviation unity (independent of distance). The dilution matrix ε_{ij} takes values 1 or 0, and a nonzero entry appears with probability p_{ij}, where $p_{ij} \sim r_{ij}^{-2\sigma}$, with $r_{ij} = (L/\pi)\sin(\pi|i-j|/L)$ representing the geometric distance between the spins. The power σ is a key parameter of the model. To avoid the probability of placing a bond being larger than 1, a short-distance cutoff is applied, and thus we take

$$p_{ij} = 1 - \exp(-A/r_{ij}^{2\sigma}), \qquad z = \sum_{i=1}^{L-1} p_{iL}. \tag{2}$$

The constant A is determined numerically by fixing the average coordination number z. Note that this model has the same long-range interactions on average, $[J_{ij}^2]_{\rm av} \sim 1/r_{ij}^{2\sigma}$, as in KY but has only $Lz/2$ bonds rather than $L(L-1)/2$. Hence the linear scaling of the CPU time for one MCS.

As in the fully connected case [7], by varying σ one can tune the model in Eq. (1) from the infinite-range to the short-range universality class. For $0 < \sigma \leq 1/2$ the model is in the infinite-range universality class in the sense that the parameter A vanishes for $N \to \infty$, and for $\sigma = 0$ it corresponds to the Viana-Bray model [11]. For $1/2 < \sigma \leq 2/3$ the model describes a mean-field long-range spin glass, corresponding—in the analogy with short-range systems—to a short-range model in a dimension above the upper critical dimension $d \geq d_u = 6$ [12]. For $2/3 < \sigma \leq 1$ the model has non-mean-field critical behavior with a finite transition temperature T_c. For $\sigma \geq 1$, the transition temperature is zero. We are interested in finite-range models which have a nonzero T_c, i.e., $1/2 < \sigma \leq 1$.

A rough correspondence between a value of σ in the long-range 1D model and the value of a space dimension d in a short-range model can be obtained from

$$d = \frac{2 - \eta(d)}{2\sigma - 1}, \tag{3}$$

where $\eta(d)$ is the critical exponent η for the short-range model, which is zero in the mean-field regime. Equation (3) has the following required properties: (i) $d \to \infty$ corresponds to $\sigma \to 1/2$, (ii) the upper critical dimension $d_u = 6$ corresponds to $\sigma_u = 2/3$, and (iii) the lower critical dimension, which is where $d_l - 2 + \eta(d_l) = 0$, corresponds to $\sigma_l = 1$. For example, in 3D, $\eta = 0.384(9)$ [13], and thus the corresponding exponent is $\sigma \simeq 0.90$.

In this study, we set the average coordination number to $z_{\rm av} = 6$ and use site-dependent random fields h_i chosen from a Gaussian distribution with zero mean $[h_i]_{\rm av} = 0$ and standard deviation $[h_i^2]_{\rm av}^{1/2} = H_{\rm R}$. The latter has the advantage that we can perform a detailed test for equilibration of the data when using Gaussian-distributed interactions [7,14] (see below).

To determine the existence of an AT line, we compute the two-point finite-size correlation length [5,15,16]. For this we start by determining the wave-vector-dependent spin-glass susceptibility given by

$$\chi_{\rm SG}(k) = \frac{1}{L}\sum_{i,j}[(\langle S_i S_j\rangle_T - \langle S_i\rangle_T\langle S_j\rangle_T)^2]_{\rm av} e^{ik(i-j)}, \tag{4}$$

where $\langle \ldots \rangle_T$ denotes a thermal average and $[\ldots]_{\rm av}$ an average over the disorder. To avoid bias, each thermal average is obtained from a separate copy of the spins, so we simulate four copies at each temperature. The correlation length is given by [7]

$$\xi_L = \frac{1}{2\sin(k_m/2)}\left[\frac{\chi_{\rm SG}(0)}{\chi_{\rm SG}(k_m)} - 1\right]^{1/(2\sigma-1)}, \tag{5}$$

where $k_m = 2\pi/L$ is the smallest nonzero wave vector compatible with the boundary conditions. According to finite-size scaling,

$$\begin{aligned} \xi_L/L &\sim \mathcal{X}[L^{1/\nu}(T - T_c)], & \sigma > 2/3, \\ \xi_L/L^{\nu/3} &\sim \mathcal{X}[L^{1/3}(T - T_c)], & 1/2 < \sigma \leq 2/3, \end{aligned} \tag{6}$$

with $\nu = 1/(2\sigma - 1)$ in the mean-field regime [8]. Hence, if there is a transition at $T = T_c$, data for ξ_L/L ($\xi_L/L^{\nu/3}$ in the mean-field region) for different system sizes L should cross at T_c.

We also present data for $\chi_{\rm SG} \equiv \chi_{\rm SG}(0)$, which has the finite-size scaling form

$$\begin{aligned} \chi_{\rm SG} &\sim L^{2-\eta}\mathcal{C}[L^{1/\nu}(T - T_c)], & \sigma > 2/3, \\ \chi_{\rm SG} &\sim L^{1/3}\mathcal{C}[L^{1/3}(T - T_c)], & 1/2 < \sigma \leq 2/3. \end{aligned} \tag{7}$$

Hence, curves of $\chi_{\rm SG}/L^{2-\eta}$ ($\chi_{\rm SG}/L^{1/3}$ in the mean-field

TABLE I. Parameters of the simulations for different field strengths $H_{\rm R}$ and exponents σ. $N_{\rm sa}$ is the number of samples, $N_{\rm sw}$ is the total number of Monte Carlo sweeps, $T_{\rm min}$ is the lowest temperature simulated, and N_T is the number of temperatures used in the parallel tempering method for each system size L. The last column shows the parameter A [Eq. (2)] fixing $z_{\rm av} = 6$ neighbors.

σ	$H_{\rm R}$	L	$N_{\rm sa}$	$N_{\rm sw}$	$T_{\rm min}$	N_T	A
0.60	0.10	128	8000	8192	0.480	46	0.994 58
0.60	0.10	256	8000	32 768	0.480	46	0.903 63
0.60	0.10	512	5000	131 072	0.480	46	0.838 27
0.60	0.10	1024	5000	524 288	0.480	46	0.789 26
0.60	0.10	2048	4500	65 536	1.393	26	0.751 40
0.75	0.00	128	5000	32 768	0.300	50	1.711 41
0.75	0.00	256	5000	32 768	0.300	50	1.642 89
0.75	0.00	512	5000	524 288	0.300	50	1.598 59
0.75	0.00	1024	2900	2 097 152	0.300	50	1.569 03
0.75	0.00	2048	1000	2 097 152	0.480	46	1.548 92
0.75	0.00	4096	1000	65 536	1.192	31	1.535 06
0.75	0.00	8192	500	131 072	1.192	31	1.525 44
0.75	0.10	128	5000	32 768	0.480	46	1.711 41
0.75	0.10	256	5000	131 072	0.480	46	1.642 89
0.75	0.10	512	5000	262 144	0.480	46	1.598 59
0.75	0.10	1024	5000	524 288	0.480	46	1.569 03
0.75	0.10	2048	2800	524 288	0.710	39	1.548 92
0.85	0.10	128	6000	16 384	0.300	50	2.394 85
0.85	0.10	256	6000	65 536	0.300	50	2.348 67
0.85	0.10	512	6800	524 288	0.300	50	2.321 89
0.85	0.10	1024	2500	2 097 152	0.300	50	2.305 92

regime) should also intersect. This is particularly useful for long-range models since η is given by the naive expression $2 - \eta = 2\sigma - 1$ *exactly.*

As discussed in KY, for the simulations to be in equilibrium with Gaussian fields and bonds, the following equality must hold:

$$U(\hat{q}_l, q) = -\frac{1}{T}\left[\frac{N_b}{L}(1 - \hat{q}_l)\right]_{\rm av} - \frac{H_{\rm R}^2}{T}(1 - q), \qquad (8)$$

where $q = L^{-1}\sum_i[\langle S_i\rangle_T^2]_{\rm av}$ is the spin overlap, $\hat{q}_l = N_b^{-1}\sum_{i,j}\varepsilon_{ij}\langle S_iS_j\rangle_T^2$ is the link overlap of a given sample, and N_b is the number of nonzero bonds of the sample. To speed up equilibration, we use the parallel tempering (exchange) Monte Carlo method [17,18]. Simulations are performed at zero field, as well as at $H_{\rm R} = 0.1$, a value considerably smaller than $T_c(H_{\rm R} = 0)$ for the values of σ studied. For details, see Table I.

Results.—We start by showing in Fig. 1(a) data for ξ_L/L against T for $\sigma = 0.75$ in zero field, for several system sizes. The data intersect cleanly at $T_c \simeq 1.50$, indicating a transition at that point; see Eq. (6). The inset shows $\chi_{\rm SG}/L^{2-\eta}$ using the exact value $\eta = 1.5$.

In contrast to Fig. 1(a), which shows the expected zero-field transition for $\sigma = 0.75$, Fig. 1(b) shows no intersections in a small field $H_{\rm R} = 0.1$ [approximately 0.067 of the zero-field T_c shown in Fig. 1(a)]. Thus there is no AT line for $\sigma = 0.75$, except possibly for even smaller values of the field. Note that $\sigma = 0.75$ is in the non-mean-field regime ($2/3 < \sigma < 1$). Whereas the data for $\sigma = 0.75$

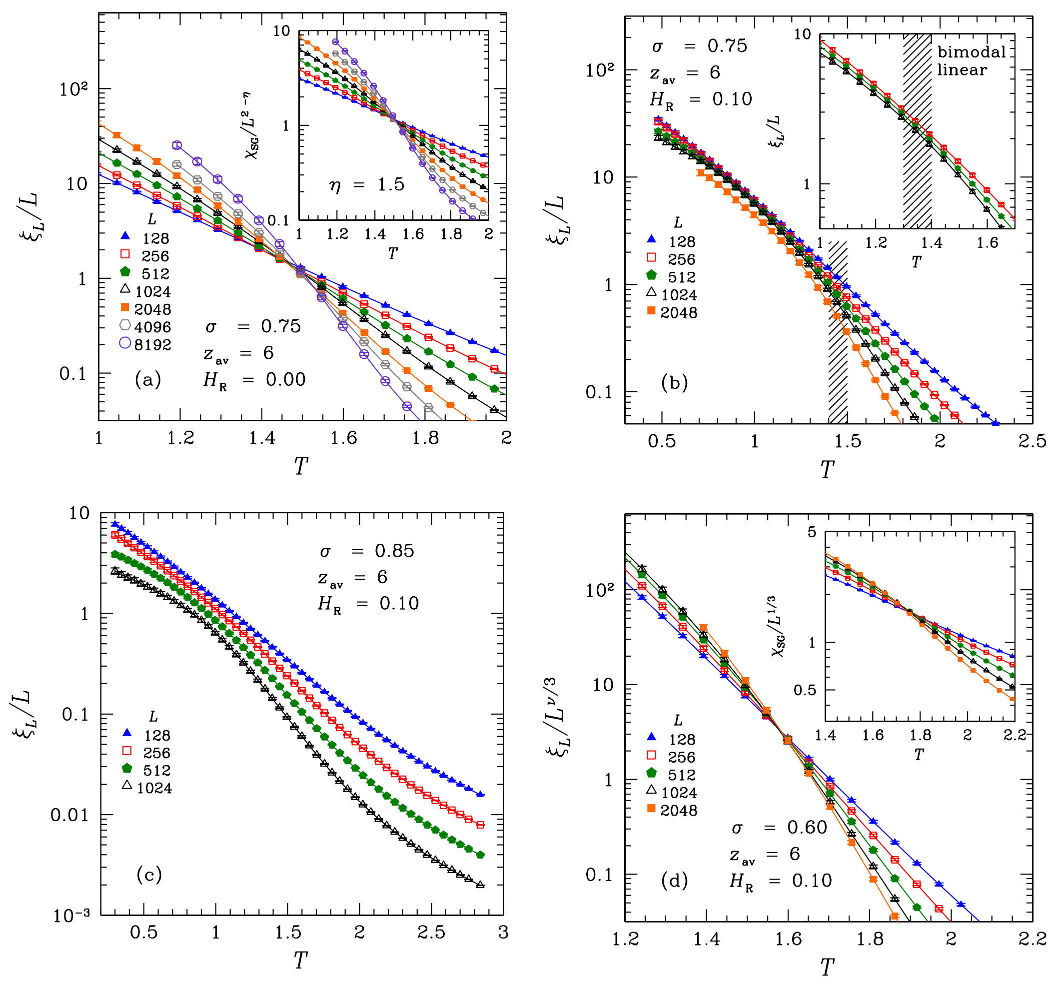

FIG. 1 (color online). (a) Finite-size correlation length divided by L as a function of T for different sizes for $H_{\rm R} = 0$ and $\sigma = 0.75$ (non-mean-field region). The inset shows $\chi_{\rm SG}/L^{2-\eta}$ using the exact value $\eta = 3 - 2\sigma = 1.5$. In both cases, the data cross, indicating a phase transition at zero field. (b) The same as (a) but for $H_{\rm R} = 0.1$. The absence of an intersection down to low T shows that there is no transition in a field [the shaded area corresponds to $T_c(H_{\rm R} = 0)$]. The inset shows data for a bimodal ($\pm J$) distribution of bonds, as used in Ref. [19], for sizes $L = 256$–1024 on a linear topology. While Ref. [19] find a finite-temperature transition (shaded area in the inset), we see no sign of it. The absence of a transition is even more clear in (c), where we show data as in (b) but for $\sigma = 0.85$, i.e., deeper into the non-mean-field regime. In (d) we show data for the correlation length divided by $L^{\nu/3}(= L^{5/3})$ as a function of T for different sizes for $H_{\rm R} = 0.1$ and $\sigma = 0.60$ (in the mean-field region). The inset shows $\chi_{\rm SG}/L^{1/3}$. The intersections show that there is a transition in a field, i.e., an AT line for this value of σ.

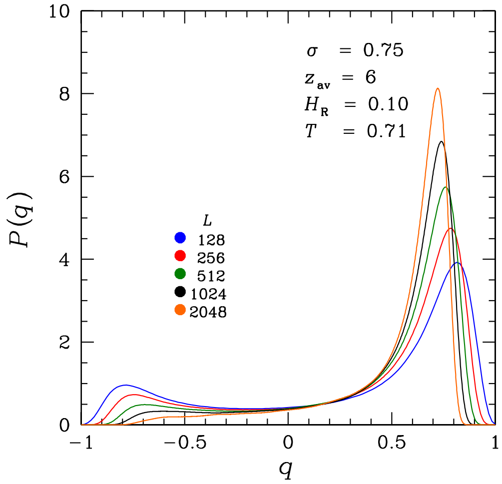

FIG. 2 (color online). Distribution of the spin overlap q for $\sigma = 0.75$, $T = 0.71$, and $H_R = 0.1$. Even for the largest L studied, there is a tail which extends into the negative-q region.

for small sizes merge, and it is only for the *larger* sizes that the data do not even meet, for $\sigma = 0.85$—deeper in the non-mean-field regime—even the data for small sizes do not meet at any temperature down to $T = 0.30$; see Fig. 1(c).

For comparison, we also show data in the mean-field regime where an AT line is expected to occur [7]. For $\sigma = 0.60$ and $H_R = 0.1$, there is a clear intersection; see Fig. 1(d). The temperature of the intersections is slightly different in the two cases, about 1.60 for $\xi_L/L^{5/3}$ and about 1.75 for $\chi_{SG}/L^{1/3}$, suggesting finite-size effects, possibly due to long negative tails in the spin overlap distribution; see Fig. 2 and Ref. [19].

We note that very recent work by Leuzzi *et al.* [19] comes to a different conclusion. Using Eq. (1) with bimodally distributed disorder, they find a transition in a field in the non-mean-field regime, in particular, for $\sigma = 0.75$ and $H_R = 0.1$, where we do not find a transition; see Fig. 1(b). We have no explanation for this discrepancy. We have done several checks, including developing two versions of the code independently and verifying that they give the same results. Furthermore, we have simulated the model with the same bimodal disorder and geometry as used in Ref. [19], as well as the same field and σ values, finding no signature of a transition [see the inset in Fig. 1(b)].

Summary.—Our conclusion, based on numerical results, is that there is an ''upper critical dimension'' close to 6 for the AT line, in agreement with KY. This conclusion is distinct from RSB theory [2], which predicts an AT line in any space dimension with a zero-field transition, and the droplet picture [3,4], according to which there is no AT line in any finite dimension. Of course, the numerical data cannot rule out a transition at *extremely* small fields.

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Note added in proof.—We have recently learned [20] that there was an error in the analysis of Ref. [19] and that their results for $\sigma = 0.75$ are now much closer to ours.

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