Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Point

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We extend the definition of "spectral dimension" d_s (usually defined for fractal and lattice geometries) to theories in spacetimes with anisotropic scaling. We show that in gravity with dynamical critical exponent z in $D+1$ dimensions, the spectral dimension of spacetime is $d_s = 1 + \frac{D}{z}$. In the case of gravity in $3+1$ dimensions with $z = 3$ in the UV which flows to $z = 1$ in the IR, the spectral dimension changes from $d_s = 4$ at large scales to $d_s = 2$ at short distances. Remarkably, this is the behavior found numerically by Ambjørn *et al.* in their causal dynamical triangulations approach to quantum gravity.

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The idea that the effective spacetime dimension can change with the scale is not new. One simple thing that can happen as we probe spacetime at shorter distances is that extra dimensions can emerge. The fact that our macroscopic Universe appears, to a good approximation, four dimensional is then viewed as a result of course graining. Such extra dimensions can be of the Kaluza-Klein type [1–3], or our observed Universe can be the boundary of a higher-dimensional space [4–7], or a higher codimension brane, perhaps with additional warping of the full geometry. Another intriguing possibility is that the nature of the four macroscopic dimensions themselves may qualitatively change with scale. The poor short-distance behavior of general relativity has often been interpreted as an indication that something radical must happen to spacetime at short distances. It has been speculated that at some characteristic scale (often related to the Planck scale), the smooth geometry of spacetime could be replaced by a discrete structure, or exhibit fractal behavior or a stringy generalization of geometry, or that the short-distance nature of spacetime might be nongeometric altogether. This picture is further supported by string theory, in which the macroscopic spacetime (or at least space) can often be viewed as an emergent concept.

In recent numerical simulations of lattice quantum gravity in the framework of causal dynamical triangulations (CDT) [8–10], an interesting phenomenon has been observed: The effective spacetime dimension is four at large scales, but changes continuously to two at short distances [11]. The four-dimensional nature of spacetime at large scales indicates a good long-distance continuum limit. However, the interpretation of the change in dimension at shorter scales is not clear. Perhaps the geometry undergoes a dynamical dimensional reduction, or develops a foamy structure at short distances. The lattice methods of dynamical triangulations do not offer enough analytical control over the dynamics of geometry. It would be desirable to compare this against an analytical tool in a continuum framework. The CDT approach to quantum gravity has one distinguishing feature: The triangulations are restricted to conform to a preferred causal structure, given by a preferred foliation by slices of constant time. This is motivated by the desire to maintain causality and leads to the suppression of baby universes, which—when present—are believed to be responsible for the pathological branched-polymer scaling in the continuum limit. This preferred causal structure of the CDT framework is reminiscent of the symmetries in the recently proposed Lifshitz phase of quantum gravity [12,13], which is defined in the path-integral framework and exhibits anisotropic scaling at short distances. The degree of anisotropy is measured by the dynamical critical exponent z, which changes from $z = 3$ in the UV to the relativistic value $z = 1$ in the IR. In this Letter, we present some evidence suggesting that the CDT approach to lattice gravity may in fact be a lattice version of the quantum gravity at a Lifshitz point. Using the same definition of dimension as in the CDT approach [11], we show that in the continuum framework of [13] the effective dimension of the Universe flows from four at large distances to two at short distances, reproducing the lattice results of [11].

In principle, there are many different ways of defining the dimension of a fluctuating geometry. Here we follow [11], and consider a measure of dimension which has proven useful in discretized approaches to quantum gravity in low dimensions: the "spectral dimension" of spacetime. The idea is simple: Spectral dimension of a geometric object $\mathcal{M}$ is the effective dimension of $\mathcal{M}$ as seen by an appropriately defined diffusion process (or a random walker) on $\mathcal{M}$. Such a process is characterized by the probability density $\rho(\mathbf{w}, \mathbf{w}'; \sigma)$ of diffusion from point $\mathbf{w}$ to $\mathbf{w}'$ in $\mathcal{M}$, in diffusion time σ, subjected to the initial condition $\rho(\mathbf{w}, \mathbf{w}'; 0) = \delta(\mathbf{w} - \mathbf{w}')$. The *average return probability* $P(\sigma)$ is obtained by evaluating $\rho(\mathbf{w}, \mathbf{w}'; \sigma)$ at $\mathbf{w} = \mathbf{w}'$ and averaging over all points $\mathbf{w}$ in $\mathcal{M}$. The *spectral dimension* of $\mathcal{M}$ is then defined as

$$d_s = -2 \frac{d \log P(\sigma)}{d \log \sigma}. \quad (1)$$

For example, in the case of $\mathcal{M} = \mathbf{R}^d$ with the flat

Euclidean metric, we obtain

$$\rho(\mathbf{w}, \mathbf{w}'; \sigma) = \frac{e^{-(\mathbf{w}-\mathbf{w}')^2/4\sigma}}{(4\pi\sigma)^{d/2}}. \quad (2)$$

In this case, (1) gives $d_s(\mathbf{R}^d) = d$, which simply reproduces the topological dimension of the Euclidean space. The spectral dimension can be defined in a manifestly coordinate-independent way, which makes it applicable to a wide range of objects beyond smooth manifolds, including objects with fractal behavior. Indeed, systems are known for which d_s is not an integer: e.g., the spectral dimension of branched polymers [14] is $d_s = 4/3$.

The spectral dimension has been used [15–22] as one of the simplest observables probing the continuum limit in the lattice approach to quantum gravity in two dimensions. This case is relevant for the description of fluctuating world sheets in noncritical string theory. In the nonperturbative definition of the system in terms of dynamical triangulations and matrix models, the spectral dimension of world sheets has been found to be $d_s = 2$ [19], as long as the central charge of the world sheet matter sector is $c \leq 1$. Above this $c = 1$ barrier, the ensemble of fluctuating geometries is believed to collapse to a branched-polymer phase. This expectation has been further confirmed by the measurement of the spectral dimension in [15], yielding $d_s = 4/3$ above $c = 1$. Interestingly, this simplest branched-polymer phase of two-dimensional gravity is in fact the lowest member of an infinite family of multicritical phases [16], parametrized by $m = 2, 3, \ldots$, and with spectral dimensions

$$d_s = 2m/(2m-1). \quad (3)$$

In [11], the spectral dimension of spacetime was measured in the numerical CDT approach to quantum gravity in $3+1$ dimensions, with intriguing results. At long distances, the spectral dimension found by [11] is

$$d_s = 4.02 \pm 0.1. \quad (4)$$

With the changing scale, however, the spectral dimension appears to smoothly decrease to the short-distance limit, given by [11]

$$d_s = 1.80 \pm 0.25. \quad (5)$$

This value is consistent with an effective reduction of spacetime to two dimensions at short distances.

In field theories with anisotropic scaling, the degree of spacetime anisotropy is characterized by the dynamical critical exponent z,

$$\mathbf{x} \to b\mathbf{x}, \qquad t \to b^z t. \quad (6)$$

Models with $z \neq 1$ are common in condensed matter (see, e.g., [23]). Theories of gravity with various values of z in various spacetime dimensions $D+1$ were introduced in [12,13]. The case of Yang-Mills theory with $z = 2$ was discussed in [24]. For power-counting renormalizability of gravity in $3+1$ dimensions, we need $z = 3$ at short distances [13] (see also [25]). A theory of gravity in $3+1$ dimensions with $z = 3$ was presented in [13]. The field content consists of the spatial metric g_{ij}, together with the lapse and shift variables N_i and N. The theory is invariant under foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}(M)$ of spacetime, which take the coordinate form $\tilde{x}^i = \tilde{x}^i(t, x^j)$ and $\tilde{t} = \tilde{t}(t)$. The action is given by

$$S = \frac{2}{\kappa^2} \int dt d^3\mathbf{x} \sqrt{g} N \{K_{ij}K^{ij} - \lambda (K_i^i)^2 - \mathcal{V}\}. \quad (7)$$

Here $K_{ij} = (1/2N)(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ is the extrinsic curvature of the preferred time foliation. The first two terms in (7) represent the covariant kinetic term, of second order in time derivatives, with κ and λ two dimensionless couplings undetermined by the $\mathrm{Diff}_{\mathcal{F}}(M)$ symmetries. The potential term $\mathcal{V}$ in (7) is a local function of g_{ij} and $\partial_k g_{ij}$ independent of $\dot{g}_{ij}$. Unlike the kinetic term which is universal, the precise form of $\mathcal{V}$ depends on the desired value of z. For example, general relativity requires $\mathcal{V} \propto R - 2\Lambda$ (and $\lambda = 1$ for full spacetime diffeomorphism invariance), implying the relativistic value $z = 1$.

In condensed matter, a particularly interesting class of models with $z \neq 1$ satisfies an additional condition of "detailed balance." Those models are intimately related to a Euclidean theory in one lower dimension. In the case of gravity in $3+1$ dimensions, this condition means that $\mathcal{V} \sim (\delta W/\delta g_{ij})^2$, with W the action of a gravity theory in three dimensions. (The square is performed with the appropriate De Witt metric; see [12,13] for details.) The $z = 3$ gravity introduced in [13] is described by (7) with $\mathcal{V}_{\mathcal{C}} = \frac{\kappa^4}{16w^4} C_{ij}C^{ij}$, where $C^{ij} = \epsilon^{ik\ell}\nabla_k(R^j_\ell - \frac{1}{4}\delta^j_\ell R)$ is the Cotton tensor, and w is a dimensionless coupling. Since $C_{ij} = 0$ follows from the variation of the Chern-Simons action $W = (1/w^2)\int(\Gamma \wedge d\Gamma + \frac{2}{3}\Gamma \wedge \Gamma \wedge \Gamma)$, the theory satisfies detailed balance. This condition can be explicitly broken by adding diffeomorphism invariant terms of sixth order in spatial derivatives, such as $\nabla_i R_{jk} \nabla^i R^{jk}$ or $R^i_j R^j_k R^k_i$, to $\mathcal{V}$. This does not change the value of z, but theories without detailed balance are more complex due to the proliferation of independent terms in the action. Luckily, the spectral dimension will turn out to be a very universal observable, sensitive only to the scaling (6) but not to the details of $\mathcal{V}$.

The global scaling transformations (6) can be generalized to the case of curved and fluctuating spacetime geometries. In [13], the local anisotropic Weyl transformations with $z = 3$ were introduced,

$$g_{ij} \to e^{2\Omega(\mathbf{x},t)} g_{ij}, \quad N_i \to e^{2\Omega(\mathbf{x},t)} N_i, \quad N \to e^{3\Omega(\mathbf{x},t)} N. \quad (8)$$

(Other values of z were discussed in [12].) These transformations represent a local version of the global anisotropic scaling (6) of flat space, adapted to the general background g_{ij}, N_i, and N. They form a closed symmetry group with the foliation-preserving diffeomorphisms $\mathrm{Diff}_{\mathcal{F}}(M)$ (see [12,13]). Since C_{ij} transforms covariantly under spatial conformal transformations, the potential term $\mathcal{V}_C$ is invariant under (8). At the special value $\lambda = 1/3$, the kinetic term is also invariant under (8).

In $z = 3$ gravity, the leading term $\mathcal{V}_C$ in $\mathcal{V}$ is of the same dimension as the kinetic term $\sim K^2$, and dominates $\mathcal{V}$ at short distances. However, the $\mathrm{Diff}_{\mathcal{F}}(M)$ symmetries allow a number of relevant terms in $\mathcal{V}$, which affect the dynamics at long distances: The theory flows to lower z, and ultimately to $z = 1$. Such relevant terms in $\mathcal{V}$ can be generated without violating detailed balance by adding two relevant terms to the three-dimensional Chern-Simons action W: the Ricci scalar R and the cosmological constant term. This turns W into the action of topologically massive gravity, and results in the modified potential

$$\mathcal{V} = \frac{\kappa^4}{16w^4} C_{ij} C^{ij} + \cdots - \frac{c^2}{2\kappa^2}(R - 2\Lambda). \tag{9}$$

[The "..." in (9) stand for terms of fourth and fifth order in spatial derivatives.] From the perspective of the $z = 3$ UV fixed point, c and Λ are relevant coupling constants of dimension two (in units of momentum). The last two terms in (9) constitute the potential $\mathcal{V}$ in general relativity. At long distances, it is natural to redefine the time coordinate to reflect the $z = 1$ scaling, by setting $x^0 = ct$. The theory in the infrared then closely resembles general relativity, with the effective Newton constant $G_N = \kappa^2/(32\pi c)$.

To compare the behavior of the spectral dimension in the lattice CDT approach [11] with our analytic approach, we must extend the definition of spectral dimension to smooth spacetimes with anisotropic scaling. What is the appropriate diffusion process to consider? The spectral dimension of the Minkowski spacetime (with $z = 1$) is measured [11] by rotating to imaginary time $t = -i\tau$; on the Euclidean space, the diffusion process is described by the probability density ρ of (2), governed by the diffusion equation $\partial\rho/\partial\sigma = (\partial^2/\partial\tau^2 + \Delta)\rho$, with $\Delta \equiv \partial_i\partial_i$ the *spatial* Laplacian. This can be naturally generalized to the case of $z > 1$. In gravity with anisotropic scaling, the time dynamics stays the same as in the $z = 1$ case; it is the spatial dynamics that changes with the changing potential $\mathcal{V}$. This suggests that the natural diffusion process at general z is governed by the anisotropic equation

$$\frac{\partial}{\partial\sigma}\rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) = \left\{\frac{\partial^2}{\partial\tau^2} + (-1)^{z+1}\Delta^z\right\}\rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma). \tag{10}$$

Indeed, both terms on the right-hand side of (10) scale the same way under the anisotropic rescaling (6). The relative sign $(-1)^{z+1}$ in (10) is determined from the requirement of ellipticity of the diffusion operator. The formula is valid for integer z, but our results below can be analytically continued to any positive real z.

The anisotropic diffusion equation (10) is solved by

$$\rho(\mathbf{x}, \tau; \mathbf{x}', \tau'; \sigma) = \int \frac{d\omega d^D\mathbf{k}}{(2\pi)^{D+1}} e^{i\omega(\tau-\tau')+i\mathbf{x}\cdot(\mathbf{x}-\mathbf{x}')-\sigma(\omega^2+|\mathbf{k}|^{2z})}. \tag{11}$$

In order to determine the spectral dimension, we only need ρ at the coincident initial and final spacetime points,

$$\rho(\mathbf{x}, \tau; \mathbf{x}, \tau; \sigma) = \int \frac{d\omega d^D\mathbf{k}}{(2\pi)^{D+1}} e^{\sigma(\omega^2+|\mathbf{k}|^{2z})} = \frac{C}{\sigma^{(1+D/z)/2}}, \tag{12}$$

with some nonzero constant C. Using (1), we obtain the spectral dimension of spacetime with anisotropic scaling,

$$d_s \equiv -2\frac{d\log P(\sigma)}{d\log\sigma} = 1 + \frac{D}{z}. \tag{13}$$

This implies the central result of this Letter: In $3 + 1$ dimensions with $z = 3$, the spectral dimension (13) is equal to $d_s = 2$ at short distances. Under the influence of the relevant deformations, the theory flows to $z = 1$ in the infrared, and (13) reproduces the macroscopic value $d_s = 4$ at long distances. In (13), d_s was evaluated in a fixed classical spacetime geometry, with $g_{ij} = \delta_{ij}$, $N = 1$, and $N_i = 0$, and gives therefore the leading value of d_s in the semiclassical approximation. The definition of d_s can be generalized to the full quantum theory, by defining the covariant anisotropic diffusion operator on an arbitrary geometry, and averaging the return probability over all geometries in the path integral. However, with quantum corrections assumed small, the scale dependence of the spectral dimension will be dominated by the change in the anisotropic scaling of the classical solution, from $z = 3$ in the UV to $z = 1$ in the IR.

A nonzero cosmological constant may preclude the flat geometry from being a solution. If so, d_s will become sensitive at cosmological scales to the characteristic spacetime curvature. However, such finite-size effects will not change the effective value of d_s at intermediate scales.

We have extended the notion of spectral dimension to the continuum framework of quantum gravity with anisotropic scaling, and found that the behavior of d_s matches qualitatively the lattice results obtained by [11] in the CDT approach. This raises the intriguing possibility that the continuum limit of the causal dynamical triangulations may belong to the same universality class as the anisotropic theory of gravity [13], flowing from the anisotropic scaling with $z = 3$ in the UV to the relativistic value $z = 1$ in the IR. The possibility that the CDT lattice approach is effectively a lattice description of quantum field theory of gravity with anisotropic scaling presented in [12,13] is further supported by the symmetries imposed in the two frameworks. As reviewed above, theories of gravity with anisotropic scaling are invariant under foliation-preserving diffeomorphisms; the spacetime manifold is equipped with a preferred causal structure, compatible with anisotropic scaling [see Fig. 1(a)]. On the other hand, the novelty of the CDT approach to lattice gravity is that the sum is performed over lattice geometries with a preferred "causal structure" [Fig. 1(b)]. It is this extra condition on the discretizations which changes favorably the continuum limit, and prevents the collapse of the partition sum to a branched-polymer phase. It is plausible that the continuum limit of the lattice sum automatically identifies a mechanism leading to its UV completion in the minimal way

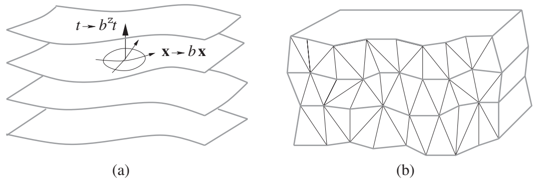

FIG. 1. (a) The preferred foliation by time slices in the continuum approach of gravity with anisotropic scaling, and (b) a characteristic lattice configuration in the CDT approach.

compatible with the preferred foliation, in terms of gravity with anisotropic scaling and $z = 3$ at short distances.

The short-distance lattice value (5) of the spectral dimension is consistent within the margin of error with $z = 3$. However, the mean value is closer to d_s with $z = 4$. Theories with $z = 4$ in $3 + 1$ dimensions satisfying the detailed balance condition were discussed in [13]: They are constructed from the three-dimensional action W containing terms up to quadratic in the Ricci tensor. (Such models of three-dimensional gravity have recently been discussed in [26].) Reasons why gravity with $z = 4$ might be desirable in $3 + 1$ dimensions were discussed in [13].

Even though the main focus of this Letter is on gravity in $3 + 1$ dimensions, our result (13) for the spectral dimension of spacetime with anisotropic scaling is general, with possible applications to quantum gravity in other dimensions. For example, it is intriguing that the spectral dimensions (3) observed in the multicritical branched-polymer phases of discretized two-dimensional gravity can be reproduced by continuum theories in $1 + 1$ dimensions with anisotropic scaling and the integer multicritical values of the dynamical exponent $z = 2m - 1$.

The spectral dimension also plays a prominent role in the thermal behavior of systems with anisotropic scaling. Simple scaling arguments show that the free energy of free massless fields at the Lifshitz point with dynamical critical exponent z scales with temperature as $F \sim T^{1+D/z} = T^{d_s}$. Notably, when $D = z$ (the critical dimension of gravity with anisotropic scaling), the behavior of the free energy $F \sim T^2$ is the same as in a relativistic conformal theory in $1 + 1$ dimensions. This scaling has been seen before [27], in the ensemble of free strings formally extrapolated above the Hagedorn temperature T_H. An example of anisotropic gravity with $z = 9$ in $9 + 1$ dimensions can be obtained by following the logic of [13]: Starting with $W \sim \int \omega_9$ [where $\omega_9 = \Gamma \wedge (d\Gamma)^4 + \cdots$ is the Chern-Simons 9-form] and setting $\mathcal{V} = (\delta W/\delta g_{ij})^2$ leads to a theory with detailed balance in $9 + 1$ dimensions with $z = 9$, whose high-temperature behavior at the free fixed point matches the scaling found in [27] in superstring theory above T_H.

In conclusion, we have demonstrated that even for smooth spacetime geometries, the spectral dimension does not have to match the topological dimension. The discrepancy can simply result from anisotropic scaling, compatible with a preferred causal structure of spacetime. This suggests an alternative interpretation of the dynamical reduction of spacetime at short distances [11] observed in the lattice approach to quantum gravity: This behavior does not have to indicate a change in the topological dimension of spacetime, or a foamy structure in which the four macroscopic dimensions result from coarse graining over topologically complicated two-dimensional geometries. Instead, the behavior of [11] can simply be explained by anisotropic scaling of space and time at short distances, keeping the topology of spacetime four dimensional and its geometry smooth and topologically trivial.

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