Distance Dependence of Entanglement Generation via a Bosonic Heat Bath

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Within a generalized Caldeira-Leggett model, we analyze the conditions under which a bosonic heat bath can entangle two microscopic quantum systems at a distance r. We find that the attainable entanglement is extremely distance-sensitive. Significant entanglement can only be achieved if the systems are within a *microscopic* distance that is of order of the cutoff wavelength λ of the system-bath interaction. At larger distances, the maximal entanglement is exponentially suppressed with a decay length of order λ. We conclude that entanglement generation via a heat bath is not suitable for entangling remote objects.

Establishing and preserving quantum-mechanical entanglement [1] between two remote microscopic physical systems is an experimentally challenging undertaking. Generally speaking, the difficulties arise from the fact that the systems need to significantly interact with each other (in order to build up an entangled common state), and at the same time, the two systems must be thoroughly shielded from interaction with external degrees of freedom (in order to preserve the entangled state against decoherence [2]). In most physical situations, these two requirements appear to be contradicting to each other and therefore can be only partially fulfilled.

During the last years, a fresh look at this entangling dilemma has emerged from theoretical work on the dynamics of entanglement in open systems, notably from the work of Braun [3,4] and Benatti *et al.* [5,6]. It has been shown that under suitable conditions, two two-level systems [3–5,7,8] or two harmonic oscillators [6,9] can become entangled by mere interaction with a common bosonic heat bath, without any direct interaction between the microscopic systems. In such a situation, the coupling to the heat bath has two relevant effects: it leads to decoherence, as it usually does, but it also mediates an effective interaction between the systems. When the latter one is strong enough to overcompensate the decohering effect, the coupling to the heat bath may eventually lead to entangled microscopic systems. Entanglement generation via environmental modes is, needless to say, a sophisticated mechanism. Its theoretical analysis therefore necessarily has to rely on idealizing assumptions, and it is still not clear to which extent these assumptions can be met in real systems.

We pursued research that especially addresses the role of the spatial distance r between the microscopic systems on the entangling mechanism. In doing so, we *fully* account for *dissipative* system-bath interactions by investigating an exactly solvable model along the lines of Ullersma [10], and Caldeira and Leggett [11]. Existing studies [4,8] of the distance dependence are either confined to a *dissipation-free* spin-boson model or treat dissipation on a perturbative level only. In contrast to the comparatively moderate power-law dependence observed in [4,8], here we find significant entanglement between the systems only if the distance r does not much exceed the cutoff wavelength λ of the system-bath interaction. At larger distances, $E_{\max}$, the maximum attainable logarithmic negativity (as a measure of entanglement [12]), decreases exponentially with a decay length of order λ. We argue that λ will be typically of order of the spatial extension of the microscopic systems and thus conclude that entanglement generation via a heat bath is limited to truly microscopic distances only.

In our model, the two remote microscopic quantum systems are represented by two identical harmonic oscillators located on a line at positions $x_{1/2} = \pm r/2$. The choice of harmonic oscillators makes the model exactly solvable. Nevertheless, we still expect it to capture the basic physical behavior of any system with a discrete spectrum (cf. [13]). The oscillators, henceforth called the *system oscillators*, have mass m and frequency ω_0, and P_1, Q_1 and P_2, Q_2 denote their canonical variables. They are coupled to an extended, one-dimensional heat bath consisting of symmetric ($\propto \cos kx$) and antisymmetric ($\propto \sin kx$) harmonic modes of wave numbers $k > 0$ and frequencies $\omega_k = ck$, c being the velocity of sound/light. Let $p_k^{s/a}$, $q_k^{s/a}$ denote their respective canonical variables. We consider a bilinear system-bath interaction H_I, where the two oscillators locally couple to symmetric and antisymmetric bath modes in the same manner,

$$H_I = \sum_k g_k (Q_1 + Q_2) q_k^s \cos\frac{kr}{2} + g_k (Q_1 - Q_2) q_k^a \sin\frac{kr}{2}.$$

The coupling strengths g_k may be characterized as usual by a spectral function $J(\omega) := \sum_k \frac{g_k^2}{2m_k \omega_k} \delta(\omega - \omega_k)$. Here, we assume $J(\omega)$ to be linear for small ω with a Drude cutoff, $J(\omega) = \frac{2m\gamma}{\pi} \omega \frac{\Omega^2}{\Omega^2 + \omega^2}$, leading to ohmic damping with a damping constant γ. Typically, the cutoff frequency $\Omega \equiv 2\pi c/\lambda$ is *not* some intrinsic frequency of the bath; rather, it

will be determined by the physics of the system-bath coupling. Then, since generally $|g_k|$ markedly declines when $|k|^{-1}$ falls below the spatial extension l of the microscopic systems, a good order of magnitude estimate is $\lambda \sim l$, meaning that $\Omega \sim 2\pi c/l$.

We also include a counter-term $V_c = \sum_k \frac{g_k^2}{2m_k\omega_k^2}[Q_1^2 + 2Q_1Q_2\cos(kr) + Q_2^2]$ in the total Hamiltonian. Its purpose is twofold: first, it removes the frequency renormalization caused by the coupling to the bath [11]; second, it ensures that the Quantum Langevin Equations (QLE) which we are going to derive below will only contain retarded couplings between the oscillators.

The dynamics of the system oscillators can be approached by means of the Heisenberg equations of motions for their coordinates Q_1, Q_2. Following the analysis in [14], they can be written as two coupled QLE,

$$\ddot{Q}_1(t) + \omega_0^2 Q_1(t) + \frac{d}{dt}\int_0^t dt'[\Gamma_0(t-t')Q_1(t') + \Gamma_r(t-t')Q_2(t')] = B_1(t) \quad (1)$$

and a similar equation where 1 and 2 are interchanged. Here, we introduced a distance d dependent damping kernel $\Gamma_d(t) = \gamma\Omega(e^{-\Omega|t-d|} + e^{-\Omega|t+d|})$, and bath operators

$$B_{1/2}(t) = \sum_k \tilde{g}_k \cos\frac{kr}{2}e^{i\omega_k t}b_k^\dagger \pm \tilde{g}_k\sin\frac{kr}{2}e^{i\omega_k t}a_k^\dagger + \text{H.c.},$$

where $\tilde{g}_k = (\hbar g_k^2/m_k\omega_k m^2)^{1/2}$, and $b_k^\dagger$, b_k and $a_k^\dagger$, a_k are creation and annihilation operators of a symmetric and antisymmetric bath mode k. Note that the operators $B_{1/2}(t)$ evolve freely in time; the backaction of the two oscillators on the bath modes is solely contained in the memory terms in the QLE. The QLE also have a clear classical interpretation: the two oscillators are subjected to friction with a damping constant γ, they are coupled via a bath-mediated retarded interaction, and they are exposed to stochastic forces $B_{1/2}(t)$. Without V_c, the QLE also would exhibit terms proportional to $Q_1(t)Q_2(t)$, corresponding to an instantaneous, direct coupling of the two oscillators. In principle, the appearance of such term is possible because our model does not obey Lorentz invariance. Nevertheless, here we are interested in the bath-mediated coupling of the oscillators, and therefore eliminated the direct couplings by adding V_c to the system Hamiltonian.

The formal solution of the QLE is simple, once they are written in the form

$$\dot{\mathbf{y}}(t) + Z\mathbf{y}(t) + \frac{d}{dt}\int_0^t dt' \mathcal{C}(t-t')\mathbf{y}(t') = \mathbf{B}(t), \quad (2)$$

where $\mathbf{y} = (Q_1, Q_2, \dot{Q}_1, \dot{Q}_2)$, $\mathbf{B} = (0, 0, B_1, B_2)$, and Z and $\mathcal{C}(t)$ are 4×4 matrices whose definitions become obvious by comparison of Eq. (2) with the original QLE. Then, the solution $\mathbf{y}(t)$ of Eq. (2) for initial $\mathbf{y}(0)$ and inhomogeneity $\mathbf{B}(t)$ is

$$\mathbf{y}(t) = \mathcal{G}(t)\mathbf{y}(0) + \int_0^t dt'\mathcal{G}(t-t')\mathbf{B}(t'), \quad (3)$$

where the Green's function $\mathcal{G}(t)$ solves the homogeneous part of Eq. (2). Its Laplace transform $\hat{\mathcal{G}}(s) = [s + Z + s\hat{\mathcal{C}}(s)]^{-1}$ can be calculated analytically.

Correlations and entanglement in the two oscillator system can be studied on the basis of the oscillator's dimensionless covariance matrix C,

$$C_{lm} = \langle\tilde{\mathbf{y}}_l\tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_m\tilde{\mathbf{y}}_l\rangle_{\rho_s} \equiv \text{tr}[(\tilde{\mathbf{y}}_l\tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_m\tilde{\mathbf{y}}_l)\rho_s],$$

where ρ_s is the joint state of the system oscillators. The vector $\tilde{\mathbf{y}}$ is obtained from $\mathbf{y}$ by multiplying the first and second entry with $(m\omega_0/\hbar)^{1/2}$, and the third and forth entry with $(m/\hbar\omega_0)^{-1/2}$. Assuming that at time $t = 0$, the total state factorizes in an initial oscillator state ρ_s and a thermal state ρ_T of the bath, the temporal evolution of the covariance matrix follows with Eq. (3) to be

$$C(t) = \mathcal{G}(t)C(0)\mathcal{G}(t)^\dagger$$
$$+ \int_0^t dt' \int_0^t dt'' \mathcal{G}(t-t')\mathcal{K}(t'-t'')\mathcal{G}(t-t'')^\dagger.$$

Here, $C(0)$ is the covariance matrix of the initial oscillator state ρ_s. The matrix $\mathcal{K}(t) = 2m\langle\mathbf{B}(t)\mathbf{B}(0)^\dagger\rangle_{\rho_T}/\hbar\omega_0$ contains the correlations of the bosonic fields $B_{1/2}$. Its non-vanishing entries are $\mathcal{K}_{34}(t) = \mathcal{K}_{43}(t)$, equal to

$$\frac{8\gamma}{\pi\omega_0}\int_0^\infty d\omega\,\omega\frac{\Omega^2}{\Omega^2+\omega^2}\coth\frac{\omega}{2T}\cos\omega t\cos\omega r,$$

and two diagonal elements $\mathcal{K}_{33}(t) = \mathcal{K}_{44}(t)$, which are given by the same expression, but with $r = 0$.

We quantify the amount of entanglement of the two oscillators by the logarithmic negativity E. In case of a Gaussian state ρ_s of the oscillators, E can be conveniently determined from the correlation matrix C as follows: First, one applies a time-reversing operation [15] on the second oscillator, according to which the covariance matrix transforms to $C_{lm} \to \tilde{C}_{lm} = (-1)^{\delta_{l4}+\delta_{m4}}C_{lm}$. Then, the symplectic eigenvalues λ_1, λ_2 of $\tilde{C}$ yield the logarithmic negativity as $E = -\sum_{j=1}^2 \log_2\min(1, \lambda_j)$ [12]. In this way, the entanglement dynamics of the two oscillators follows from the temporal evolution of the correlation matrix C, provided that the oscillator state is Gaussian for all times. This is the case when we restrict ourself to Gaussian initial states since this property is conserved under the dynamics of the quadratic Hamiltonian. Note that for Gaussian states a vanishing logarithmic negativity is equivalent to separability [15].

Having outlined our model and the methods we have used, let us now present our results. Our main interest is the generation of entanglement from an initially separable oscillator state $\rho_s(0)$ via the coupling with the bosonic bath. Since there are no reasons for certain initial separable states being preferred to other ones, here we present only results where initially the system oscillators are in their

ground state. The bath is assumed to be initially in a thermal state ρ_T of temperature T. Thus, the total state is Gaussian, and we can determine the logarithmic negativity E as a function of time as outlined above. In the following, we will mainly show numerical data demonstrating the characteristic dependence of the entanglement generation on distance r, cutoff frequency Ω, damping constant γ, and temperature T. Generally, we measure distances in units of c/ω_0, frequencies in units of ω_0, and temperature in units of $\hbar\omega_0/k_B$.

First, we consider the entanglement of the two oscillators at large times. For any finite distance r, one finds $\mathcal{G}(t) \to 0$ for $t \to \infty$, meaning that the initial oscillator state becomes irrelevant at large times. Hence, the asymptotic covariance matrix is $C_\infty = \int_0^\infty dt' \int_0^\infty dt'' \mathcal{G}(t')\mathcal{K}(t'-t'')\mathcal{G}(t'')^\dagger$. The time integrations together with the oscillating factors in $\mathcal{K}(t'-t'')$ represent Laplace transformations, which eventually result in a single ω integral over terms containing the factor $|\hat{\mathcal{G}}(i\omega)|^2$. The remaining integration over ω can be easily performed numerically. Figure 1 shows the asymptotic logarithmic negativity E as a function of the distance r between the oscillators. Clearly, the entanglement decreases with distance and drops to zero at rather small critical distances d_0. The dependence of d_0 on the inverse cutoff frequency Ω for different temperatures can be seen in the inset of Fig. 1. For $\Omega \gtrsim \omega_0$, we find the critical distance d_0 to be inversely proportional to the cutoff frequency, $d_0 \approx ac/\Omega$, where a is a coefficient of order unity (at $T = 0$) that decreases with increasing temperature. The distance d_0 is rather insensitive to the actual value of the damping constant γ. For instance, the critical distance of $d_0 = 0.151$ (in units of c/ω_0) at $\gamma = \omega_0$ and $T = 0$ just changes to 0.12 or 0.17 when the damping is increased or lowered by a factor of 10, respectively.

Now we consider how the logarithmic negativity develops in time. Determining the time-dependent covariance
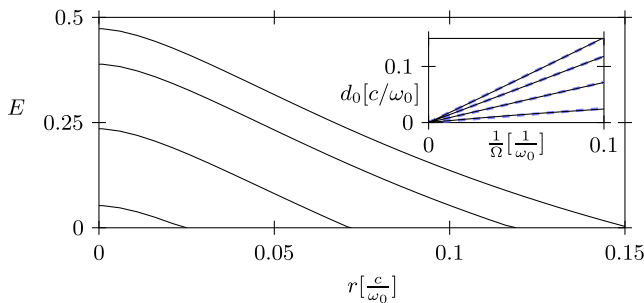
matrix $C(t)$ involves an inverse Laplace transformation, which we performed numerically using Durbin's formula [16]. Results for vanishing and three nonvanishing distances r below the critical distance d_0 are shown in Fig. 2. All curves show a characteristic peak at short times within which the logarithmic negativity reaches its maximum value E_{max}. After its decay, the logarithmic negativity slowly recovers in an oscillatory manner to its asymptotic value, where the frequency of the oscillation is approximately $\omega_0/2$. The oscillations decay rather slowly with time because the relative coordinate $Q_1 - Q_2$ of the two oscillators is weakly damped for the small distances r under consideration. This behavior does not change much for distances slightly above d_0. However, at larger distances $r > 0.18c/\omega_0$, the logarithmic negativity does not recover at all but remains zero for all later times.

Focusing on the short time behavior of E, the initial peak actually resolves in two peaks, as shown in Fig. 3. The first peak appears immediately after switching on the interaction at times t less than r/c. Since bosons cannot have been exchanged between the two system oscillators within this time span, we attribute this peak to entanglement that had been already present in the bath [17]. Switching on the interaction might immediately transfer part of that entanglement to the oscillators. This behavior can be addressed by a short time expansion of $C(t)$, which, at zero temperature, eventually results in

$$E(t) \approx \frac{4}{\ln 2}\frac{\gamma}{\omega_0}\{e^{-(r\Omega/c)}\Omega t - \alpha_{\Omega t}(\Omega t)^2 + \mathcal{O}(\Omega t)^3\}, \quad (4)$$

with an Ωt dependent $\alpha_{\Omega t} \approx 0.2937 - \frac{1}{\pi}\ln\Omega t$. From this expansion, we find that for $r \gtrsim c/\Omega$ width and height of the first peak are exponentially suppressed in the parameter $r\Omega/c$ by factors less than $\sim \exp(-r\Omega/c)$ and $\sim \exp(-2r\Omega/c)$, respectively.

The second peak in the logarithmic negativity is delayed by a little bit more than r/c, which suggests that it refers to entanglement due to exchange of bosons. Its height decreases monotonically with distance r and, in fact, reaches zero at a relatively small distance d_1 that is constrained by the inverse cutoff frequency. At zero temperature and

FIG. 1 (color online).   Asymptotic entanglement of the system oscillators measured in logarithmic negativity E as a function of distance r (in units of c/ω_0) for temperatures $T = 0, 0.1, 0.2,$ and $0.3 \times \hbar\omega_0/k_B$ (upper to lower curves), damping constant $\gamma = \omega_0$, and cutoff frequency $\Omega = 10\omega_0$. E drops to zero at a rather small critical distance $d_0 \lesssim c/\Omega$, which for the above temperatures is proportional to Ω^{-1} (cf. inset). The fitted straight lines (dashed lines) have slopes $a = 1.51, 1.18, 0.72, 0.25$.
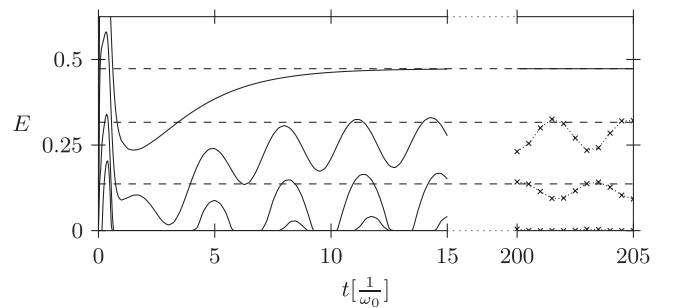

FIG. 2.   Logarithmic negativity E as function of time (in units of $1/\omega_0$) for distances $r = 0, 0.05, 0.1,$ and $0.15 \times c/\omega_0$ (upper to lower curves) below the critical distance d_0, $T = 0$, $\Omega = 10\omega_0$, and $\gamma = \omega_0$. Dashed lines represent asymptotic values.
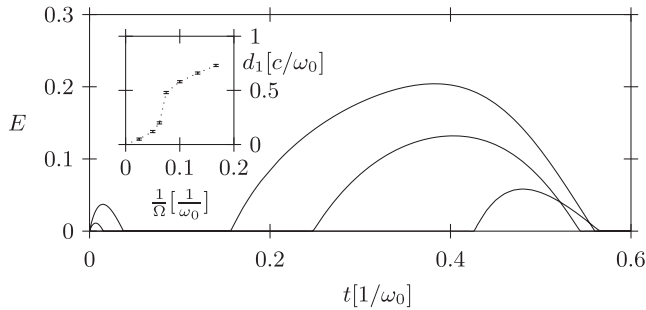
FIG. 3.   Short time behavior of E for $r = 0.15$, 0.2, and 0.4 (in units of c/ω_0, top to bottom). The initial peak of $E(t)$ visible in Fig. 2 resolves in two peaks. The first peak is exponentially suppressed in $2r\Omega/c$. The second peak is delayed by approximately r/c. Its height decreases with r and vanishes for distances $r \gtrsim d_1$. The inset shows d_1 as function of the inverse cutoff frequency $1/\Omega$ (in units of $1/\omega_0$).

damping $\gamma = \omega_0$, numerical data show $d_1(\Omega) \lesssim 6.0c/\Omega$ (cf. inset of Fig. 3). We expect that the actual value of the damping constant γ has only minor influence on d_1 (as like on the distance d_0), since numerical data as well as Eq. (4) show that in first approximation γ scales only the amplitude of $E(t)$.

We conclude that generally for distances r significantly larger than c/Ω, the logarithmic negativity $E(t)$ reaches its total maximum $E_{\max}$ within an exponentially short time $t_0 \lesssim \exp(-r\Omega/c)/\Omega$ and then vanishes for all times $t \gtrsim 2t_0$. Moreover, at these distances, the maximum value $E_{\max}$ is exponentially suppressed in $2r\Omega/c$.

To summarize, by analyzing the time-dependent logarithmic negativity of two oscillators coupled to a bosonic bath, we found strong evidence that the entanglement mechanism under consideration is limited to rather small distances r of order of c/Ω, i.e., to distances of order of the cutoff wavelength λ. In practice, this length corresponds to the spatial extension of the microscopic systems to be entangled. At larger distances, the maximum achievable logarithmic negativity is exponentially suppressed in, roughly, r/λ. We believe that this behavior is characteristic for bath-mediated entanglement in general since there seem to be no features of the investigated oscillator model which would make it special for entanglement. In fact, the general picture outlined here is fully supported by results that we obtained for an alternative two-spin-boson model [18]. Having said this, one may summarize our findings by stating that generally two objects can only be efficiently entangled via the interaction with a heat bath if they are in the immediate vicinity of each other.

It might appear puzzling that the environment quickly and strongly entangles with the two oscillators (which, after all, is the origin of the ubiquitous phenomenon of decoherence), while the two oscillators for their own remain essentially disentangled (if they are remote from each other). The reason behind this strongly asymmetric behav-

ior is the large asymmetry in the (effective) Hilbert space dimensions of the participating systems: *few* oscillator states interact with a *continuum* of bath states. Assuming that the *generic* state of the joint system is well represented by a randomly chosen state of the joint system, it follows from [19] that for dimensional reasons, the bath is strongly entangled with each oscillator, while the system oscillators on their own remain separable. Thus, our analysis particularly demonstrates that under the actual dynamics—generated by a standard bilinear system-bath interaction—a nongeneric initial state rapidly evolves to a generic one. Interestingly, the considered interaction fails to produce this effect if the distance r becomes less or of the order of the cutoff wavelength λ, as evidenced in significant entanglement of the system oscillators in this case.

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