Cosmology from an Anti-de Sitter–Schwarzschild Black Hole via Holography

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We derive the equations of cosmological evolution from an anti-de Sitter–Schwarzschild black hole via holographic renormalization with appropriate boundary conditions.

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The effect of extra dimensions demanded by string theory on cosmology has been extensively investigated. In a popular scenario [1], one considers a three-brane within a higher-dimensional bulk. The cosmological evolution of the brane [2] is equivalent to its motion within the bulk space [3]. The bulk may be occupied by a black hole, or a more complicated (or unknown) solution to the Einstein field equations [4,5]. The second possibility seems more appropriate for a cosmological setup and allows for energy exchange between the brane and the bulk [6].

Without the brane-world assumption, the connection of the AdS/CFT correspondence to cosmology is elusive. Our aim is to understand how cosmological evolution emerges in the context of AdS/CFT [7]. We shall show that the equations of cosmological evolution emerge via holographic renormalization [8], even when one starts from a static anti-de Sitter–Schwarzschild black hole, provided the boundary conditions are chosen appropriately.

In general, one starts with a solution to the bulk Einstein equations with a negative cosmological constant and proceeds to compute the properties of the dual gauge theory at strong coupling on the anti-de Sitter (AdS) boundary. One usually fixes the geometry of the boundary by adopting Dirichlet boundary conditions. However, in order to obtain cosmological evolution, the boundary geometry must remain dynamical. Changing the boundary conditions in order to accommodate a dynamical boundary metric may lead to fluctuations of the bulk metric which are not normalizable [9]. It was recently shown that such fears are unfounded, and the boundary geometry can be dynamical if one correctly introduces boundary counterterms needed in order to cancel infinities [10].

We shall concentrate on an AdS-Schwarzschild black hole in five dimensions, which is a solution to the Einstein field equations with a negative cosmological constant ($\Lambda_5 = -6/l^2$),

$$R_{AB} = \frac{2}{l^2} g_{AB}, \qquad (1)$$

where $A, B = 0, 1, 2, 3, 4$. The metric can be written in static coordinates as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2, \quad f(r) = r^2 + k - \frac{\mu}{r^2}, \qquad (2)$$

where $k = +1, 0, 1$ for spherical, flat, and hyperbolic horizons, respectively. We set $l = 1$ for simplicity.

The Hawking temperature and mass of the hole are, respectively,

$$T_H = \frac{2r_+^2 + k}{2\pi r_+}, \qquad M = \frac{3V_k}{16\pi G_5} r_+^2(r_+^2 + k), \qquad (3)$$

where V_k is the volume of the three-dimensional space Σ_k spanned by Ω_k, r_+ is the radius of the horizon, and G_5 is Newton's constant in the bulk.

The Einstein equations (1) are obtained by varying the bulk action $I_{\mathcal{M}}$. This is a five-dimensional Einstein-Hilbert action on $\mathcal{M}$ with a cosmological term. It also includes a Gibbons-Hawking boundary term, as well as boundary counterterms needed to render the system finite. In addition to the black hole solution in the bulk, the variation of the action in general yields a boundary term

$$\delta I_{\mathcal{M}} = \frac{1}{2} \int_{\partial\mathcal{M}} d^4x \sqrt{-\det g^{(0)}} T_{\mu\nu}^{(\mathrm{CFT})} \delta g^{(0)\mu\nu}, \qquad (4)$$

where $g_{\mu\nu}^{(0)}$ ($\mu, \nu = 0, 1, 2, 3$) is the boundary metric and $T_{\mu\nu}^{(\mathrm{CFT})}$ the stress-energy tensor of the dual conformal field theory (CFT) on the AdS boundary $\partial\mathcal{M}$.

If one adopts Dirichlet boundary conditions that fix $g_{\mu\nu}^{(0)}$, the additional term (4) vanishes. As we are interested in keeping $g_{\mu\nu}^{(0)}$ dynamical, following [10], we shall adopt *mixed* boundary conditions instead (see also [11]). To define them, we shall introduce a boundary action consisting of two terms,

$$I_{\partial\mathcal{M}} = I_{\partial\mathcal{M}}^{(\mathrm{EH})} + I_{\partial\mathcal{M}}^{(\mathrm{matter})}. \qquad (5)$$

The first term is the Einstein-Hilbert action in four dimensions

$$I_{\partial\mathcal{M}}^{(\mathrm{EH})} = -\frac{1}{16\pi G_4} \int_{\partial\mathcal{M}} d^4x \sqrt{-\det g^{(0)}} (\mathcal{R} - 2\Lambda_4), \qquad (6)$$

where G_4 (Λ_4) is Newton's constant in the four-dimensional boundary and $\mathcal{R}$ the four-dimensional Ricci scalar (constructed from $g^{(0)}_{\mu\nu}$). The second term is an unspecified action for matter fields,

$$I^{(\mathrm{matter})}_{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} d^4x\sqrt{-\det g^{(0)}}\mathcal{L}^{(\mathrm{matter})}. \tag{7}$$

The matter fields reside on the boundary and have no bulk duals.

To the variation of the bulk action (4) we must now add the variation of the new boundary action $\delta I_{\partial\mathcal{M}}$, given by (5), under a change in the boundary metric. Demanding that the sum vanish

$$\delta I_{\mathcal{M}} + \delta I_{\partial\mathcal{M}} = 0 \tag{8}$$

leads to two possibilities: (a) Dirichlet boundary conditions, i.e., fixed $g^{(0)}_{\mu\nu}$, or (b) *mixed* boundary conditions:

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}\mathcal{R} - \Lambda_4 g^{(0)}_{\mu\nu} = 8\pi G_4(T^{(\mathrm{CFT})}_{\mu\nu} + T^{(\mathrm{matter})}_{\mu\nu}). \tag{9}$$

The latter, which we shall adopt, are nothing but the Einstein field equations in four dimensions. Moreover, the variation of the boundary matter action (7) under a change in the matter fields leads to the standard four-dimensional matter field equations.

In [10], it was shown that a general form of $I_{\partial\mathcal{M}}[g^{(0)}]$ leads to a sensible theory with normalizable metric fluctuations. Matter fields were not considered. However, the framework may be extended to include boundary matter fields. If one integrates over them in the path integral, an effective action is obtained as a functional of the boundary metric $g^{(0)}$. For this effective action, the discussion in [10] is applicable.

In order to understand the AdS/CFT correspondence, it is useful to write the metric in terms of Fefferman-Graham coordinates [12]. Define z through

$$\frac{dz}{z} = -\frac{dr}{\sqrt{f(r)}} \tag{10}$$

which gives (with an appropriate integration constant)

$$z^4 = \frac{16}{k^2 + 4\mu}\frac{r^2 + \frac{k}{2} - r\sqrt{f(r)}}{r^2 + \frac{k}{2} + r\sqrt{f(r)}}. \tag{11}$$

This equation may be inverted to give

$$r^2 = \frac{\alpha + \beta z^2 + \gamma z^4}{z^2} \tag{12}$$

where

$$\alpha = 1, \qquad \beta = -\frac{k}{2}, \qquad \gamma = \frac{k^2 + 4\mu}{16}. \tag{13}$$

The metric (2) reads

$$ds^2 = \frac{1}{z^2}\left[dz^2 - \frac{(1 - \gamma z^4)^2}{1 + \beta z^2 + \gamma z^4}dt^2 + (1 + \beta z^2 + \gamma z^4)d\Omega^2_k\right]. \tag{14}$$

The energy of the dual conformal field theory on the AdS boundary is found through holographic renormalization. For a metric in the form

$$ds^2 = \frac{1}{z^2}[dz^2 + g_{\mu\nu}dx^\mu dx^\nu] \tag{15}$$

where

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + z^2 g^{(2)}_{\mu\nu} + z^4 g^{(4)}_{\mu\nu} + \cdots \tag{16}$$

the stress-energy tensor of the CFT is [8]

$$\langle T^{(\mathrm{CFT})}_{\mu\nu}\rangle = \frac{1}{4\pi G_5}\Big\{g^{(4)} - \frac{1}{4}g^{(2)}g^{(0)}g^{(2)} + \frac{1}{4}\mathrm{tr}(g^{(2)}(g^{(0)})^{-1})g^{(2)} - \frac{1}{8}\{[\mathrm{tr}(g^{(2)}(g^{(0)})^{-1})]^2 - \mathrm{tr}(g^{(2)}(g^{(0)})^{-1})^2\}g^{(0)}\Big\}_{\mu\nu}. \tag{17}$$

Applying this general expression to our metric (14), we obtain the energy density and pressure, respectively,

$$\langle T^{(\mathrm{CFT})}_{tt}\rangle = 3\langle T^{(\mathrm{CFT})}_{ii}\rangle = \frac{3\gamma}{4\pi G_5} \tag{18}$$

on the static Einstein universe $\mathbb{R}\times\Sigma_k$ with metric

$$ds_0^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -dt^2 + d\Omega^2_k. \tag{19}$$

Notice that the total energy $E = \langle T^{(\mathrm{CFT})}_{tt}\rangle V_k$ is larger than the mass of the black hole [Eq. (3)] by a constant (Casimir energy) in the case of a curved horizon ($k \neq 0$). The two quantities agree for flat horizons ($k = 0$). We shall show that the additional piece can be understood by a change of the vacuum state from Minkowski vacuum to the conformal vacuum. [Note that the curved metrics (19) can be conformally mapped on the Minkowski space.]

For general cosmological applications, instead of the static boundary considered above [with metric (19)], we need a boundary with the form of a Robertson-Walker spacetime

$$ds_0^2 = g^{(0)}_{\mu\nu}dx^\mu dx^\nu = -d\tau^2 + a^2(\tau)d\Omega^2_k, \tag{20}$$

on which to apply holographic renormalization. To this end, we need to choose a different foliation away from the black hole, consisting of hypersurfaces whose metric is asymptotically of the form (20). Therefore, we need to make a change of coordinates $(t, r) \to (\tau, z)$ and bring the black hole metric (2) in the form

$$ds^2 = \frac{1}{z^2}[dz^2 - \mathcal{N}^2(\tau, z)d\tau^2 + \mathcal{A}^2(\tau, z)d\Omega^2_k], \tag{21}$$

where $\mathcal{N}(\tau, z) \to 1$ and $\mathcal{A}(\tau, z) \to a(\tau)$ as we approach the boundary $z = 0$. Comparison with the static case [Eq. (12)] suggests the ansatz

$$\mathcal{A}^2 = \alpha(\tau) + \beta(\tau)z^2 + \gamma(\tau)z^4, \tag{22}$$

where $\alpha(\tau)$, $\beta(\tau)$, $\gamma(\tau)$ are functions to be determined.

The function $\mathcal{N}$ is constrained by the τz component of the Einstein equations (1) to be of the form

$$\mathcal{N} = \frac{\dot{\mathcal{A}}}{\delta(\tau)}. \tag{23}$$

Agreement with the boundary metric (20) then fixes

$$\alpha(\tau) = a^2(\tau), \qquad \delta(\tau) = \dot{a}(\tau). \tag{24}$$

The diagonal components of the Einstein equations collectively yield

$$\beta = -\frac{\dot{a}^2 + k}{2}. \tag{25}$$

The rest of the Einstein equations are satisfied provided

$$\beta\dot{\beta} = 2(\dot{\alpha}\gamma + \alpha\dot{\gamma}), \tag{26}$$

which may be integrated to give

$$\gamma = \frac{(\dot{a}^2 + k)^2 + 4\mu}{16a^2}, \tag{27}$$

where we fixed the integration constant by comparing with the static case [Eq. (13)].

Thus, the metric (21) is uniquely specified. It agrees with the black hole metric (2) provided

$$\frac{(r')^2}{f(r)} - f(r)(t')^2 = z^{-2} \qquad \frac{r'\dot{r}}{f(r)} - f(r)t'\dot{t} = 0$$
$$\frac{\dot{r}^2}{f(r)} - f(r)\dot{t}^2 = -\mathcal{N}^2 z^{-2} \qquad r = \mathcal{A}z^{-1}. \tag{28}$$

The last equation fixes $r(\tau, z)$. Two of the other three equations may then be used to determine the derivatives $\dot{t}$ and t'. We obtain

$$\dot{t} = -\frac{\dot{\mathcal{A}}r'}{f\dot{a}}, \qquad t' = -\frac{\dot{a}}{zf}. \tag{29}$$

In fact, these expressions satisfy all three equations. One can then verify the consistency of the system (28) by calculating the mixed derivative $\dot{t}'$ using each of the two equations (29), and showing that the two expressions match. Upon integration, we obtain a unique function $t(\tau, r)$, up to an irrelevant constant. General explicit expressions are unwieldy and will not be reported here. For example, for pure AdS space in Poincaré coordinates ($\mu = 0$, $k = 0$), we obtain

$$t(\tau, z) = -\frac{2\dot{a}z^2}{4a^2 - \dot{a}^2z^2} + \int^\tau \frac{d\tau'}{a(\tau')}, \tag{30}$$

so that at the boundary ($z = 0$), the coordinate t reduces to conformal time $\int^\tau d\tau'/a(\tau')$, while it receives corrections as we move into the bulk. This is generally the case. However, general explicit expressions are not needed in order to extract physical results because we already know the explicit form of the metric in the new coordinates [Eqs. (21)–(27)].

The stress-energy tensor of the dual CFT on the cosmological boundary (20) is determined via holographic renormalization [Eq. (17)]. We obtain the energy density and pressure, respectively,

$$\langle (T^{(\mathrm{CFT})})_{\tau\tau}\rangle = \frac{3}{64\pi G_5}\frac{(\dot{a}^2 + k)^2 + 4\mu}{a^4}$$
$$\langle (T^{(\mathrm{CFT})})^i_i\rangle = \frac{(\dot{a}^2 + k)^2 + 4\mu - 4a\ddot{a}(\dot{a}^2 + k)}{64\pi G_5 a^4}, \tag{31}$$

where no summation over i is implied. (i can be chosen in any spatial direction due to isotropy.) We deduce the conformal anomaly

$$g^{(0)\mu\nu}\langle T^{(\mathrm{CFT})}_{\mu\nu}\rangle = -\frac{3\ddot{a}(\dot{a}^2 + k)}{16\pi G_5 a^3}. \tag{32}$$

The above results can also be derived directly from the standard expressions for the energy and pressure of a gauge-theory plasma in Minkowski space through an entirely four-dimensional calculation. To this end, we observe that the Robertson-Walker metric (20) is conformally equivalent to the flat Minkowski metric. The vacuum expectation value (31) is calculated in the *conformal vacuum*. We may instead calculate the VEV in the *Minkowski vacuum*, in which case we obtain the static plasma result. The two VEVs are related through [13]

$$\langle T^{(\mathrm{CFT})}_{\mu\nu}\rangle|_{\mathrm{RW}} = \frac{1}{a^4}\langle T^{(\mathrm{CFT})}_{\mu\nu}\rangle|_{\mathrm{Minkowski}} + \mathbf{a}H^{(1)}_{\mu\nu} + \mathbf{b}H^{(3)}_{\mu\nu}, \tag{33}$$

with

$$H^{(1)}_{\mu\nu} = 2(\nabla_\nu\nabla_\mu - g^{(0)}_{\mu\nu}\nabla^2)\mathcal{R} - \frac{1}{2}g^{(0)}_{\mu\nu}\mathcal{R}^2 + 2\mathcal{R}\mathcal{R}_{\mu\nu}$$
$$H^{(3)}_{\mu\nu} = \frac{1}{12}\mathcal{R}^2\delta^\nu_\mu - \mathcal{R}^{\rho\sigma}\mathcal{R}_{\rho\mu\sigma}{}^\nu. \tag{34}$$

The curvature tensor $\mathcal{R}^\mu{}_{\nu\rho\sigma}$ is derived from the Robertson-Walker metric $g^{(0)}_{\mu\nu}$ of Eq. (20). The coefficients **a** and **b** are related to the conformal anomaly and depend on the field content of the CFT. For a theory with n_B spin-0 bosons, n_F fermions, and n_V vector fields, we have

$$\mathbf{a} = -\frac{n_B + 3n_F - 18n_V}{1080(4\pi)^2}, \qquad \mathbf{b} = \frac{n_B + \frac{11}{2}n_F + 62n_V}{180(4\pi)^2}. \tag{35}$$

For the $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory, we have $n_B = 6N^2$, $n_F = 4N^2$, and $n_V = N^2$. Therefore,

$$\mathbf{a} = 0, \qquad \mathbf{b} = \frac{N^2}{32\pi^2}. \tag{36}$$

After some algebra, Eq. (33) is seen to agree with (31), with $G_5 \sim N^{-2}$.

The temperature on the boundary may also be understood by comparing with the case of a static plasma. For a plasma in the static Einstein universe (19), the temperature coincides with the Hawking temperature T_H of the black hole (3). Since the RW metric (20) is conformally equivalent to (19), the conformal factor being a^2, the Euclidean proper time period of thermal Green functions in the RW metric scales as a. As a result, the temperature T of the universe (inversely proportional to the period) scales as a^{-1}. It coincides with T_H when $a = 1$.

Finally, the boundary conditions (9) yield the equation of cosmological evolution [5]

$$H^2 + \frac{k}{a^2} - \frac{\Lambda_4}{3} = \frac{1}{16\pi G_5}\left[\left(H^2 + \frac{k}{a^2}\right)^2 + \frac{4\mu}{a^4}\right] + \frac{8\pi G_4}{3}\rho, \tag{37}$$

where $\rho = T_{00}^{(\mathrm{matter})}$ is the energy density of four-dimensional ''ordinary'' matter (without a bulk dual) and we have introduced the Hubble parameter $H = \dot{a}/a$. This equation has the expected form, reflecting the conformal anomaly and the presence of a radiative energy component whose energy density scales $\sim a^{-4}$.

We have discussed how the equations of cosmological evolution can be obtained in the context of the AdS/CFT correspondence. Two essential elements are necessary: (a) the choice of appropriate boundary conditions so that the boundary metric becomes dynamical, and (b) the derivation of the gravity solution in terms of coordinates such that the boundary metric has the Robertson-Walker form. We have demonstrated that the procedure correctly reproduces the expected four-dimensional cosmological behavior, starting from a static AdS-Schwarzschild five-dimensional solution. The challenge for the future is to repeat the procedure for gravity duals of more realistic theories. This requires the deviation from conformal invariance and the presence of additional fields. This procedure may lead to the understanding of the nonperturbative aspects of cosmological phase transitions. For example, the deconfinement phase transition could be discussed starting from a holographic QCD model [14].

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