Duality and the Knizhnik-Polyakov-Zamolodchikov Relation in Liouville Quantum Gravity

Bertrand Duplantier[1] and Scott Sheffield[2,*]

[1]Institut de Physique Théorique, CEA/Saclay, F-91191 Gif-sur-Yvette Cedex, France

[2]Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

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We present a (mathematically rigorous) probabilistic and geometrical proof of the Knizhnik-Polyakov-Zamolodchikov relation between scaling exponents in a Euclidean planar domain D and in Liouville quantum gravity. It uses the properly regularized quantum area measure $d\mu_\gamma = \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)} dz$, where dz is the Lebesgue measure on D, γ is a real parameter, $0 \leq \gamma < 2$, and $h_\varepsilon(z)$ denotes the mean value on the circle of radius ε centered at z of an instance h of the Gaussian free field on D. The proof extends to the boundary geometry. The singular case $\gamma > 2$ is shown to be related to the quantum measure $d\mu_{\gamma'}$, $\gamma' < 2$, by the fundamental duality $\gamma\gamma' = 4$.

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Introduction.—One of the major theoretical advances in physics over the past 30 years has been the realization in gauge theory or string theory that transition amplitudes require summing over random surfaces, which replaces traditional sums over random paths, i.e., the celebrated Feynman path integrals of quantum mechanics and quantum field theory. Polyakov [1] first understood that the summation over random Riemannian metrics involved could be represented mathematically by the now celebrated *Liouville theory of quantum gravity.*

The latter can be simply described as follows: Consider a bounded planar domain $D \subset \mathbb{C}$ as the parameter domain of the random Riemannian surface and an instance h of the Gaussian free field (GFF) on D, with Dirichlet energy $(h, h)_\nabla := (2\pi)^{-1} \int_D \nabla h(z) \cdot \nabla h(z) dz$. The quantum area is then (formally) defined by $\mathcal{A} = \int_D e^{\gamma h(z)} dz$, where dz is the standard 2D Euclidean (i.e., Lebesgue) measure and $e^{\gamma h(z)}$ the *random* conformal factor of the Riemannian metric, with a constant $0 \leq \gamma < 2$. The quantum Liouville action is then

$$S(h) = \tfrac{1}{2}(h, h)_\nabla + \lambda \mathcal{A}, \tag{1}$$

where $\lambda \geq 0$ is the so-called "cosmological constant."

Kazakov introduced the key idea of placing (critical) statistical models on random planar lattices, when exactly solving there the Ising model [2]. This anticipated the breakthrough by Knizhnik, Polyakov, and Zamolodchikov (KPZ) [3], who predicted that corresponding critical exponents (i.e., conformal weights x) of any critical statistical model in the Euclidean plane and in quantum gravity (Δ) would obey the KPZ relation [3–5]:

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta. \tag{2}$$

In the critical continuum limit, the statistical system borne by the random lattice is described by a conformal field theory (CFT) with central charge $c \leq 1$, which *fixes* the value $\gamma = (\sqrt{25 - c} - \sqrt{1 - c})/\sqrt{6} \leq 2$ [3]. This provides the core continuous model of "2D quantum gravity," whose deep and manifold connections to string theory, conformal field theory, random planar lattice models, random matrix theory, and stochastic Loewner evolution (SLE) are often still conjectural (see [6–11], and references therein).

Despite its great importance for conformal field theory, and its manifold checks by explicit calculations in geometrical models on random planar lattices [12–15], the KPZ relation (2) was never proven rigorously, nor its range of validity properly defined, and not even its geometrical meaning fully understood. The aim of this Letter is to present such a proof in a minimal, yet rigorous way. In our geometrical and probabilistic approach, we start from the *critical* Liouville gravity, with action S (1) taken at $\lambda = 0$, i.e., a *free-field* action. We define a properly regularized quantum area measure, which allows for a transparent probabilistic understanding of the KPZ relation (2) for any scaling fractal set in D, as a direct consequence of the underlying *Brownian stochastic properties* of the two-dimensional GFF. We also prove the boundary analog of KPZ for fractal subsets of the boundary ∂D.

One striking and important consequence of our perspective is that KPZ appears to hold in a much broader context than the original CFT realm which relates γ to c, i.e., for *any fractal structure as measured with the quantum random measure* $e^{\gamma h(z)} dz$ *and for any* $0 \leq \gamma < 2$. For instance, it predicts that the set of Euclidean exponents x of a random or a self-avoiding walk (a $c = 0$ CFT) obey (2) with $\gamma = \sqrt{8/3}$ in pure gravity ($c = 0$) but also with $\gamma = \sqrt{3}$ on a random lattice equilibrated with Ising spins ($c = 1/2$). This *central charge mixing* yields new KPZ exponent Δ's, settling theoretically an issue raised earlier but inconclusively in numerical simulations [16,17].

Our probabilistic approach also allows us to explain the *duality* property of Liouville quantum gravity: For $\gamma > 2$, the *singular* quantum measure can be properly defined in

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terms of the *regular* γ'-quantum measure, for the dual value $\gamma' = 4/\gamma < 2$, establishing the existence of the so-called "other branch" of the γ-KPZ relation and its correspondence to standard γ'-KPZ for $\gamma' < 2$, as argued long ago by Klebanov [18–20].

An extended mathematical version of this work will appear elsewhere [21]. Several follow-up works exist, at either the rigorous level [22,23] or the heuristic one [24].

GFF circular average and Brownian motion.—Let h be a centered Gaussian free field on a bounded simply connected domain D with Dirichlet zero boundary conditions. As already remarked in Ref. [25], special care is required to make sense of the quantum gravity measure, since the GFF is a distribution and not a function (it typically oscillates between $\pm\infty$) (see, e.g., [26]).

For each $z \in D$, write $B_\varepsilon(z) = \{w : |w - z| < \varepsilon\}$. When $B_\varepsilon(z) \subset D$, write $h_\varepsilon(z)$ for the average value of h on the circle $\partial B_\varepsilon(z)$. Denote by $\rho_\varepsilon^z(y)$ the uniform density (of total mass one) localized on the circle $\partial B_\varepsilon(z)$, such that one can write the scalar product on D: $h_\varepsilon(z) = (h, \rho_\varepsilon^z) := \int_D h(y)\rho_\varepsilon^z(y)dy$. To the density ρ_ε^z is naturally associated a Newtonian potential. We define the function $f_\varepsilon^z(y)$, for $y \in D$:

$$f_\varepsilon^z(y) = -\log \max(\varepsilon, |z - y|) - \tilde{G}_z(y), \tag{3}$$

where $\tilde{G}_z(y)$ is the *harmonic* function of $y \in D$, with boundary value equal to the restriction of $-\log|z - y|$ to ∂D. By construction this $f_\varepsilon^z(y)$ satisfies Dirichlet boundary conditions and the Poisson equation $-\Delta f_\varepsilon^z = 2\pi\rho_\varepsilon^z$. This (regular) potential function is represented in Fig. 1. Integrating by parts, we immediately have the following:

$$h_\varepsilon(z) = (h, f_\varepsilon^z)_\nabla, \tag{4}$$

in terms of the Dirichlet inner product defined as

$$(f_1, f_2)_\nabla := (2\pi)^{-1} \int_D \nabla f_1(z) \cdot \nabla f_2(z) dz,$$

i.e., the interaction energy of fields associated with potentials $f_{i=1,2}$. In fact, the random variables $(h, f)_\nabla$ are zero mean Gaussian random variables for each f, with the covariance property: $\mathrm{Cov}((h, f_1)_\nabla, (h, f_2)_\nabla) = (f_1, f_2)_\nabla$. From this, we deduce the covariance of the averaged $h_\varepsilon(z)$ fields (4) on two nested circles (Fig. 1): $\mathrm{Cov}(h_\varepsilon(z), h_{\varepsilon'}(z)) = (f_\varepsilon^z, f_{\varepsilon'}^z)_\nabla$. The latter is the Newtonian interaction energy of the two circles, which, owing to the explicit potential (3), to Gauss' theorem, and to harmonicity of $\tilde{G}_z$, gives the explicit form of the covariance

$$\begin{aligned}\mathrm{Cov}\,(h_\varepsilon(z), h_{\varepsilon'}(z)) &= \mathbb{E}(h_\varepsilon(z)h_{\varepsilon'}(z)) \\ &= -\log \max(\varepsilon, \varepsilon') - \tilde{G}_z(z),\end{aligned} \tag{5}$$

with $\mathbb{E}h_\varepsilon(z) = 0$ for Dirichlet boundary conditions, and where $\tilde{G}_z(z) = -\log C(z; D)$ in terms of the so-called conformal radius C of D viewed from z, a smooth function of z. From (5) we thus get the two important variances

$$\mathrm{Var}\, h_\varepsilon(z) = -\log\varepsilon + \log C(z; D), \tag{6}$$

$$\mathrm{Var}\,[h_\varepsilon(z) - h_{\varepsilon'}(z)] = |\log\varepsilon - \log\varepsilon'|. \tag{7}$$

The interpretation of (6) and (7) is immediate: For fixed z, the Gaussian random variable $h_\varepsilon(z)$ is *one-dimensional standard Brownian motion when parameterized by time* $t := -\log\varepsilon$ [21].

Random metrics and Liouville quantum gravity.—Recall first that if N is a Gaussian random variable with mean a and variance b, then $\mathbb{E}e^N = e^{a+b/2}$. Since $\mathbb{E}h_\varepsilon(z) = 0$, we have from (6) the exponential expectation

$$\mathbb{E}\, e^{\gamma h_\varepsilon(z)} = e^{\mathrm{Var}[\gamma h_\varepsilon(z)]/2} = [C(z; D)/\varepsilon]^{\gamma^2/2}. \tag{8}$$

Since (8) ultimately *diverges* for $\varepsilon \to 0$, we are led to *regularize* Liouville quantum gravity by defining the random measure

$$d\mu_\gamma = M_\varepsilon(z)dz, \qquad M_\varepsilon(z) := \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}, \tag{9}$$

in a way similar to the so-called Wick normal ordering (see, e.g., [27]). In Ref. [21], it is shown that the limit of this regularized measure exists as $\varepsilon \to 0$, which mathematically defines Liouville quantum gravity (see also [28]).

GFF sampling and random metrics.—We now consider a measure on pairs (z, h), where h is the Gaussian free field, and, given h, the point z is chosen from the regularized quantum area measure $e^{\gamma h_\varepsilon(z)}dz$. Such a measure has the form $e^{\gamma h_\varepsilon(z)}dhdz$, where dh represents the (whole) GFF measure. Its total action is thus the quadratic combination $\frac{1}{2}(h, h)_\nabla - \gamma h_\varepsilon(z)$. Owing to (4) and to $\mathrm{Var}h_\varepsilon(z) = (f_\varepsilon^z, f_\varepsilon^z)_\nabla$, the latter can be rewritten as $\frac{1}{2}(h', h')_\nabla - \frac{\gamma^2}{2}\mathrm{Var}h_\varepsilon(z)$, with the substitution $h' := h - \gamma f_\varepsilon^z$. The probability weight involved in our random metric can therefore be written as

$$\exp[-\tfrac{1}{2}(h', h')_\nabla]\, \mathbb{E}e^{\gamma h_\varepsilon(z)}, \tag{10}$$

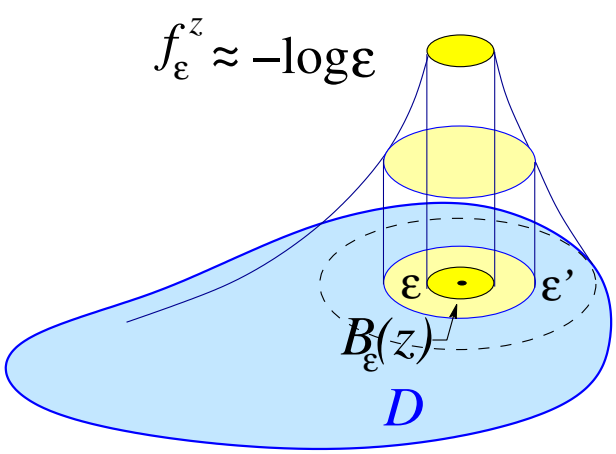

FIG. 1 (color online). Potential f_ε^z (3) created by a uniform mass distribution on the circle $\partial B_\varepsilon(z)$; its leading constant value inside the disk $B_\varepsilon(z)$ is $-\log\varepsilon$.

where the second factor is the marginal distribution density (8) of z. The meaning of (10) is that, after sampling z from its marginal distribution, the law of h *weighted* by $e^{\gamma h_\varepsilon(z)}$ is identical to that of the original GFF h' *plus* the deterministic function γf_ε^z (3).

KPZ proof.—It is shown in Ref. [21] that, when ε is small, the stochastic quantum measure (9) of the Euclidean ball $B_\varepsilon(z)$ is very well approximated by

$$\mu_\gamma(B_\varepsilon(z)) \simeq \pi \varepsilon^{\gamma Q} e^{\gamma h_\varepsilon(z)}, \tag{11}$$

where $Q_\gamma := 2/\gamma + \gamma/2$. In the simplified perspective of this work, we take (11) to be the *definition* of $\mu_\gamma(B_\varepsilon(z))$. That is, we view μ_γ as a function on balls of the form $B_\varepsilon(z)$, defined by (11), rather than a fully defined measure on D. Define then the *quantum ball* $\tilde{B}^\delta(z)$ of area δ centered at z as the (largest) Euclidean ball $B_\varepsilon(z)$ whose radius ε is chosen so that

$$\mu_\gamma(\tilde{B}^\delta(z)) = \delta. \tag{12}$$

One says that a (deterministic or random) fractal subset X of D has *Euclidean scaling exponent* x (and Euclidean dimension $2 - 2x$) if, for z chosen uniformly in D and independently of X, the probability $\mathbb{P}\{B_\varepsilon(z) \cap X \neq \varnothing\} \asymp \varepsilon^{2x}$, in the sense that $\lim_{\varepsilon\to 0} \log\mathbb{P}/\log\varepsilon = 2x$. Similarly, we say that X has *quantum scaling exponent* Δ if, when X and (z, h), sampled with weight (10) are chosen independently, we have

$$\mathbb{P}\{\tilde{B}^\delta(z) \cap X \neq \varnothing\} \asymp \delta^\Delta. \tag{13}$$

In weight (10), $h' = h - \gamma f_\varepsilon^z$ is a standard GFF, and thus its average has the characteristic property (7): $\mathcal{B}_t := h'_{\varepsilon=e^{-t}}(z)$ is standard Brownian motion in time $t = -\log\varepsilon$. Equation (3) then gives $h_\varepsilon(z) = h'_\varepsilon(z) - \gamma\log\varepsilon = \mathcal{B}_t + \gamma t$ (up to a bounded constant); i.e., $h_\varepsilon(z)$ in (11) sampled with (10) has the same law as *Brownian motion with drift*.

Equality of (11) to (12) then relates stochastically the Euclidean radius ε to the quantum area δ. This radius is given in terms of the *stopping time*

$$T_A = -\log\varepsilon_A := \inf\{t: -\mathcal{B}_t + a_\gamma t = A\}, \tag{14}$$

with the definitions $A := -(\log\delta)/\gamma > 0$ and $a_\gamma := Q_\gamma - \gamma = 2/\gamma - \gamma/2 > 0$ for $\gamma < 2$. A constant is absorbed in the choice of time origin such that $\mathcal{B}_0 = 0$.

The probability that the ball $B_{\varepsilon_A}(z)$ intersects X scales as $\varepsilon_A^{2x} = e^{-2xT_A}$. Computing its expectation $\mathbb{E}\exp(-2xT_A)$ with respect to the random time T_A will give the quantum probability (13). Consider then for any β the standard *Brownian exponential martingale* $\mathbb{E}[\exp(-\beta\mathcal{B}_t - \beta^2 t/2)] = 1$, valid for $0 \le t < \infty$. We can apply it at the stopping time T_A, when $T_A < \infty$ and where $\mathcal{B}_{T_A} = a_\gamma T_A - A$; we thus get for $2x = \beta_\gamma a_\gamma + \beta_\gamma^2/2$

$$\begin{aligned} &\mathbb{E}[\exp(-2xT_A)1_{T_A<\infty}] = \exp(-\beta_\gamma A) = \delta^{\Delta_\gamma}, \\ &\beta_\gamma(x) := (a_\gamma^2 + 4x)^{1/2} - a_\gamma, \qquad \Delta_\gamma := \beta_\gamma/\gamma. \end{aligned} \tag{15}$$

For $x = 0$, one finds in particular $\mathbb{P}(T_A < \infty) = \mathbb{E}[1_{T_A<\infty}] = 1$, since $\beta_\gamma(0) = 0$ for $a_{\gamma<2} > 0$, so that the conditioning on $T_A < \infty$ can actually be omitted. We thus obtain the expected quantum scaling behavior (13) with $\Delta = \Delta_\gamma$, which is the positive root to KPZ (2), QED.

The inverse Laplace transform $P_A(t)$ of (15), with respect to $2x$, is the probability density of $T_A = -\log\varepsilon_A$ such that $P_A(t)dt := \mathbb{P}(T_A \in [t, t + dt])$ [21]:

$$P_A(t) = \frac{A}{\sqrt{2\pi t^3}} \exp\left[-\frac{1}{2t}(A - a_\gamma t)^2\right]. \tag{16}$$

From (16), one deduces that, for A large (i.e., δ and ε small), $\frac{A}{T_A} = \frac{\log\delta}{\gamma\log\varepsilon_A}$ is concentrated in (15) near $a_\gamma + \gamma\Delta$. Reverse engineering to GFF h via (14), one finds that a point z that is typical with respect to the quantum measure is an *α-thick point* of h [29]: $\alpha := \lim_{\varepsilon\to 0} \log h_\varepsilon(z)/\log\varepsilon^{-1} = \gamma - \gamma\Delta$, for a fractal of quantum scaling dimension Δ.

Boundary KPZ.—Suppose that D is a domain with a (piecewise) linear boundary ∂D and h a GFF, now with *free* boundary conditions. For $z \in \partial D$, $h_\varepsilon(z)$ is the mean value of $h(z)$ on the *semicircle* $\partial B_\varepsilon(z) \cap D$, with variance scaling like $-2\log\varepsilon$. We define the boundary quantum measure $d\mu_\gamma^B := \varepsilon^{\gamma^2/4} e^{\gamma h_\varepsilon(z)/2} dz$, where now dz is Lebesgue measure on ∂D, with the conformal factor needed for integrating a quantum length instead of an area and a regulator such that the limit of μ_γ^B exists for $\varepsilon \to 0$ and $\gamma < 2$ [21]. For a fractal $X \subset \partial D$, we define boundary Euclidean ($\tilde{x}$) and quantum ($\tilde{\Delta}$) scaling exponents with this measure. We can repeat the analysis above, with now $h_\varepsilon(z)$ a standard Brownian motion $\mathcal{B}_{2t}$, with drift $\gamma t = -\gamma\log\varepsilon$, and prove the validity of the KPZ relation (2) for the pair $(\tilde{x}, \tilde{\Delta})$ [21], as anticipated in Ref. [11].

Liouville quantum duality.—For $\gamma > 2$, the Liouville measure (9) corresponds to the so-called "other" gravitational dressing of the Liouville potential [18–20]. The corresponding random surface is meant to be the scaling limit of random simply connected surfaces with large amounts of area cut off by small bottlenecks [30–34] (see also [35]). This surface turns out to be a treelike foam of Liouville quantum bubbles of *dual* parameter $\gamma' := 4/\gamma$ ("baby universes") connected to each other at "pinch points" and rooted at a "principal bubble" parameterized by D. A precise description requires additional machinery and will appear elsewhere. For now, we relate γ to γ' only formally.

The definition of quantum balls in (11) and (12) makes sense when $\gamma > 2$. Noting that $Q_{\gamma'} = Q_\gamma$, we have

$$\mu_{\gamma'}(B_\varepsilon(z)) = \pi\varepsilon^{\gamma' Q} e^{\gamma' h_\varepsilon(z)} = \mu_\gamma(B_\varepsilon(z))^{\gamma'/\gamma} = \mu_\gamma^{4/\gamma^2}$$

(up to an irrelevant power of π)—i.e., a γ-quantum ball of size δ has γ'-quantum size $\delta' := \delta^{4/\gamma^2}$. (Intuitively, the ball contains about a δ' fraction of the total γ'-quantum area but only a $\delta < \delta'$ fraction of the γ-quantum area because the latter also includes points on nonprincipal

bubbles.) The number of γ-quantum size-δ balls needed to cover the principal bubble D thus scales as $(\delta')^{-1} = \delta^{-4/\gamma^2}$.

From (2), the quantum scaling exponent Δ_γ in (15), when generalized to $\gamma > 2$, satisfies the *duality relation* $\Delta_\gamma - 1 = (4/\gamma^2)(\Delta_{\gamma'} - 1)$ [11,18–20]. If $X \subset D$ has scaling exponent x, then (13), established for $\gamma' < 2$, essentially says (see [21]) that the expected number $N_{\gamma'}(\delta', X)$ of γ'-quantum size-δ' balls [i.e., number $N_\gamma(\delta, X)$ of γ-quantum size-δ balls] required to cover X scales as $(\delta')^{\Delta_{\gamma'}-1} = \delta^{\Delta_\gamma - 1}$.

Brownian approach to duality.—When $\gamma > 2$, the *ε-regularized* measures $M_\varepsilon(z)dz$ (9) converge to zero. If we choose the pair (z, h) from the weighted measure $M_\varepsilon(z)dhdz$ as in (10) and consider the Brownian description (14), we find that $a_\gamma < 0$ for $\gamma > 2$, i.e., the drift term runs in a direction opposite to $A > 0$, so that $T_A = \infty$ for large A. The weighted measure is thus *singular*; i.e., there is a quantum area of at least δ *localized* at z for small enough δ. The Brownian martingale result (15) for $x = 0$ gives the probability, at a given z, for T_A to be finite:

$$\mathbb{P}(T_A < \infty) = \mathbb{E}[1_{T_A<\infty}] = \delta^{\Delta_\gamma(0)} = \delta^{1-4/\gamma^2} = \delta/\delta',$$

where $\Delta_\gamma(0) = (|a_\gamma| - a_\gamma)/\gamma = 1 - 4/\gamma^2$. For general x, (15) scales as δ^{Δ_γ}. We may define a *δ-regularized* measure $M^\delta(z)dhdz$ as $M_{\varepsilon_A}(z)dhdz$ restricted to the event $T_A < \infty$. Replacing γ with $\gamma' = 4/\gamma$ and δ with δ' has the same effect as multiplying $M^\delta(z)$ by $\delta'/\delta = \delta^{4/\gamma^2-1}$, so $\frac{\delta'}{\delta}M^\delta(z)dz$ converges to $d\mu_{\gamma'}$. This agrees with the *conditional* expectation scaling as

$$\frac{\mathbb{E}[\exp(-2xT_A)1_{T_A<\infty}]}{\mathbb{E}[1_{T_A<\infty}]} = \delta^{\Delta_\gamma} \times \frac{\delta'}{\delta} = \delta'^{\Delta_{\gamma'}}.$$

Using (15) and $a_{\gamma'} = -a_\gamma$, one obtains $\Delta_\gamma\Delta_{\gamma'} = x$, as anticipated in Ref. [11]. The typical GFF thickness $\alpha = \gamma(1 - \Delta_\gamma) = \gamma'(1 - \Delta_{\gamma'})$ is *invariant* under duality and obeys the Seiberg bound $\alpha \leq Q$ [7]; the *string susceptibility exponent* $\gamma_{\rm str} = 2 - 2Q/\gamma$ obeys the expected duality relation $(1 - \gamma_{\rm str})(1 - \gamma'_{\rm str}) = 1$ [11,18–20] (see also [30–34]). Finally, for the SLE$_\kappa$ process $\gamma = \sqrt{\kappa}$ [21], so that the Liouville $\gamma\gamma' = 4$ and SLE $\kappa\kappa' = 16$ dualities coincide.

We have established the KPZ relation for continuum Liouville quantum gravity. Outstanding open problems relate discrete models and SLE to Liouville quantum gravity, as described in Ref. [21]. We hope that they will be solved by the methods introduced here.

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