# Shifted Feedback Suppression of Turbulent Behavior in Advection-Diffusion Systems

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In spatiotemporal systems with advection, suppression of noise-sustained structures involves questions that are outside of the framework of deterministic dynamical systems control (such as Ott-Grebogi-Yorke-type methods). Here we propose and test an alternate strategy where a nonlocal additive feedback is applied, with the objective to create a new deterministic solution that becomes robust to noise. As a remarkable fact—though the needed parameter perturbations required have essentially a finite size—they turn out to be extraordinarily small in principle: $10^{-8}$ in the free-electron laser experiment presented here.

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A range of systems displays dynamically induced hypersensitivity to noise, which results in full-scale erratic fluctuations. This typically happens for wave patterns subjected to advection [1], which imposes a permanent drift of the structure. Experimental examples include turbulence in pulsed lasers [2,3], optical systems with transverse patterns [4], plasmas [5], and fluid dynamics [6].

In these systems, control or more generally suppression of erratic behaviors involves specific difficulties, linked to their high dimensionality and to the nonperturbative effect of noise. In particular, stabilization of a steady state is not a sufficient criterion for effective system control ("turbulent" behaviors are even observed in systems with a globally stable steady state [7–9]). This makes traditional methods for deterministic dynamical system control (Ott-Grebogi-Yorke-type methods [10]) not directly applicable.

In this Letter, we examine a feedback strategy requiring very small perturbations, taking advantage of the strong amplification properties (the so-called transient growth [8]) of the system, and we test its efficiency on a Ginzburg-Landau equation with advection. The feedback introduces a small nonlocal coupling between each site and another site located at a finite distance. This presents pictorial similarities with Pyragas-type schemes [11], though the dynamical processes involved are strongly different. We show that the stabilization process occurs via the creation of a new deterministic solution, and that—in the local saturation coupling case—the process can be understood in terms of convective and absolute instabilities. After the general study of the process on Ginzburg-Landau equations, we will present experimental results on the specific system which motivated this work: the UVSOR-II free-electron Laser.

To test the feedback strategy, we consider the following advection-diffusion equation, with finite size:

$$e_t(z,t) + v e_z(z,t) = e_{zz}(z,t) + Rg(\epsilon z)e(z,t) - Se(z,t) + \sqrt{\eta}\xi(z,t) + \alpha e(z+a,t), \quad (1)$$

where $e(z,t)$ is complex (as it is typically associated with the complex amplitude of a pattern). $t$ and $z$ represent time and space, $v$ the advection velocity ($v$ is supposed $>0$ in the following), $R$ the real gain term for pattern formation ($R > 0$ here). $\eta$ is the noise amplitude, and $\xi(z,t)$ is a delta-correlated white noise term. $g$ represents the spatial variation of the gain. $g(\epsilon z)$ is supposed to vary slowly with $z$ and is supercritical $[g(\epsilon z) > 0]$ in a finite region near the center. $\epsilon$ is a small parameter with $\epsilon^{-1}$ characterizing the system's size. In this Letter, we take $g(\epsilon z) = 1 - \epsilon^2 z^2$, and the saturation term $S$ is taken local in the first part of the Letter:

$$S = |e(z,t)|^2. \quad (2)$$

The feedback is applied through the term $\alpha e(z+a)$, with $\alpha$ and $a$ the gain and spatial shift parameters, respectively.

Without feedback ($\alpha = 0$), it is known [8] that noisy "turbulent" behavior typically appears when the velocity is beyond the convective-absolute threshold: $v > v_c = 2\sqrt{R}$. Beyond this point, the only stable attractor of the deterministic system (with $\eta = 0$) is the solution $e(z,t) = 0$. However, the system is subjected to transient growth, and small noise is strongly amplified when it is advected through the system (Fig. 1).

To achieve "control" (or more precisely to suppress erratic behaviors) the idea is to create a "loop" in the system, in the sense that the signal advected downstream is reinjected upstream ($a > 0$ if $v > 0$). The conjecture is

 

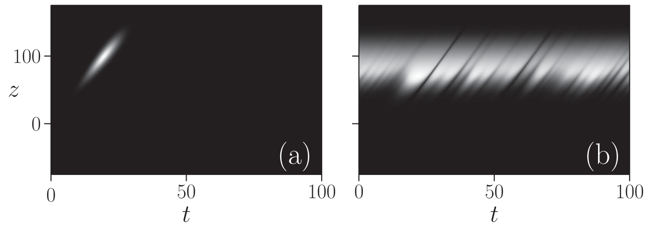


FIG. 1. Typical behavior without feedback in the convective regime ($v_c > 2\sqrt{R}$), in Eq. (1). Parameters are $\epsilon = 10^{-2}$, $R = 1$, $v = 5.4$. (a) Without noise ($\eta = 0$), and with small initial perturbation at $t = 0$ (Gaussian of amplitude $10^{-10}$) at $z = 0$. The solution tends asymptotically to zero, but passes through a state with high values [$O(1)$]. (b) With noise ($\sqrt{\eta} = 10^{-10}$), a noisy structure with $O(1)$ amplitude is observed.

that the loop can be able to change the instability character from convective to absolute, or—in other words—to create a new nonzero deterministic solution without noticeable transient growth. Numerical trials suggest that this strategy can indeed be efficient even for rather small values of the feedback coefficient [Fig. 2(a)], provided its order of magnitude exceeds the noise level $\sqrt{\eta}$. In contrast to the case without feedback, the created deterministic solution (without noise) is not significantly changed when noise is taken into account, as shown in Fig. 2(b).

In equations of the family of Eq. (1) [i.e., with local coupling and slow variations of $g(\epsilon z)$], the threshold of turbulent-regular behavior is usually near the threshold of convective-absolute instability [1,8] and to the threshold of appearance of a nonzero deterministic solution. This motivates an analytical study of the convective-absolute threshold in the presence of feedback. This consists of studying the local stability properties of the $e(z, t) = 0$ solution in the associated uniform system without noise ($\eta = 0$, $\epsilon = 0$). Since the coupling induced by the feedback is nonlocal, care must be taken in the analysis as shown by Papoff and Zambrini [12]. In the system linearized around the solution $e(z, t) = 0$, we study the response to a Dirac perturbation $\delta(z)$ at $t = 0$. To test whether the instability becomes absolute, we examine the asymptotic behavior of $e(z = 0, t)$ when $t \longrightarrow \infty$. We have

$$e(0, t) = \int_{-\infty}^{\infty} e^{f(k)t} dk, \tag{3}$$

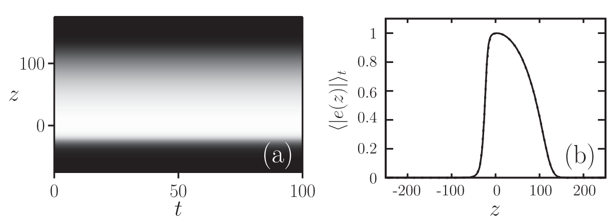


FIG. 2. With "control" ($\alpha = 10^{-4}$ and $a = 60$). (a) spatiotemporal plot in the same conditions as for Fig. 1(b). (b) temporal average of $|e(z, t)|^2$ (full line). The solution of the system without noise ($\eta = 0$) is also plotted (dotted line).

where

$$f(k) = R - k^2 - ivk + \alpha e^{ika}. \tag{4}$$

We use the classical saddle point method [12] involving extension of $f(k)$ to the complex plane $k = k_R + ik_I$. The first point consists of finding a closed integration contour with the imaginary axis, passing through saddle points ($df/dk = 0$), with steepest descents. The numerical plot of $f(k)$ easily reveals that an adequate path passes through the saddle $k_s$ defined by [$df/dk(k_s) = 0$, $\Re(k_s) = 0$]. Analytic proof remains an open question. However, once the saddle is identified, exact analytic results can be obtained, since the convective-absolute threshold occurs when $\Re[f(k_s)]$ crosses zero. This leads to the following condition for $a$ and $\alpha$:

$$\alpha = \frac{e^{\beta a}}{a}(2\beta + v), \tag{5}$$

with $\beta(<0)$ the imaginary part of $k_s$ at threshold, given by the largest solution of (note that $v > 0$, $a > 0$)

$$\beta^2 + 2\beta\left(\frac{1}{a} + \frac{v}{2}\right) + \frac{v}{a} + R = 0. \tag{6}$$

In Fig. 3(a) are displayed the convective and absolute instabilities regions [whose boundary is the solution of Eqs. (5) and (6)] for a fixed feedback level $\alpha = 10^{-4}$. It appears clearly that the desired domain of absolute instability increases with the feedback delay $a$. Furthermore, it appears also that the feedback level $\alpha$ necessary to obtain a nonzero deterministic solution (absolute instability) can be extremely small depending on the feedback delay $a$ as shown on Fig. 3(b).

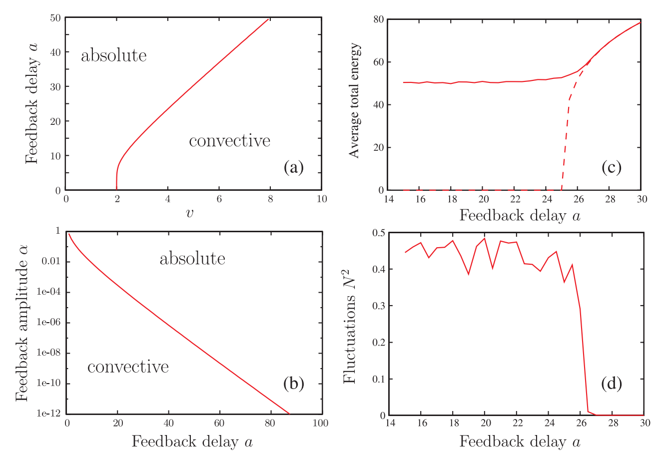


FIG. 3 (color online). (a),(b) Stability diagrams showing the new feedback-induced absolute stability regions. (c),(d) Bifurcation diagrams with noise ($\alpha = 10^{-4}$, full lines) and without noise (dashed line) versus feedback shift $a$. (c) Average total energy $\langle \int_{-\infty}^{+\infty} |e(z, t)|^2 dz \rangle$. (d) Fluctuations $N^2 = \int_{-\infty}^{+\infty} \langle |e(z, t)|^4 - \langle |e(z, t)|^2 \rangle^2 \rangle dx$, the brackets indicating temporal averaging). Parameters are $R = 1$, $v = 4$, $\epsilon = 10^{-2}$; the convective-absolute threshold is $a_c = 23.4$.

In the literature, absolute (respectively convective) instabilities are generally used as a criterion for predicting small (respectively high) sensitivity to small noise [1]. However for this criterion to hold, it is important to remember three validity conditions. The criterion is expected to hold in the limit of large system size (here $1/\epsilon \gg 1$ and $1/\epsilon \gg a$), and small noise (here $\alpha \gg \sqrt{\eta}$). As a third condition, the new created solution should not present large sensitivity to noise. We tested this last point numerically in the cases of local and global coupling.

With noise, a typical bifurcation diagram versus $a$ is presented in Figs. 3(c) and 3(d) (with $\sqrt{\eta} = 10^{-10}$ and $\alpha = 10^{-4}$). Creation of the new deterministic solution by the feedback [Fig. 3(c)] is correlated to a rapid decrease of the fluctuations [Fig. 3(d)]. Besides, the transition indeed corresponds to the absolute threshold given by Eqs. (5) and (6) (with systematic small shifts due to the finite size of the system). This indicates that the created solution does not display large transient growth when the absolute threshold is noticeably exceeded. The absolute-convective transition criterion gives quantitative information on the transition between highly and weakly noisy behavior.

An opposite situation occurs in the complex case of global saturation coupling, i.e., Eq. (1) with

$$S = \int_{-\infty}^{+\infty} |e(z,t)|^2 dx. \qquad (7)$$

Feedback is also efficient (Fig. 4) in this case and a similar interpretation applies: (i) the small feedback ($\alpha \gg \sqrt{\eta}$) creates a new stationary solution through a convective-absolute transition, and (ii) with noise, the created solution remains very close to the new stationary solution. However, dropoff of sensitivity to noise is obtained at values of $a$ that are significantly higher than the convective-absolute threshold. This theoretical complication is expected to affect generally systems with nonlocal saturation coupling,

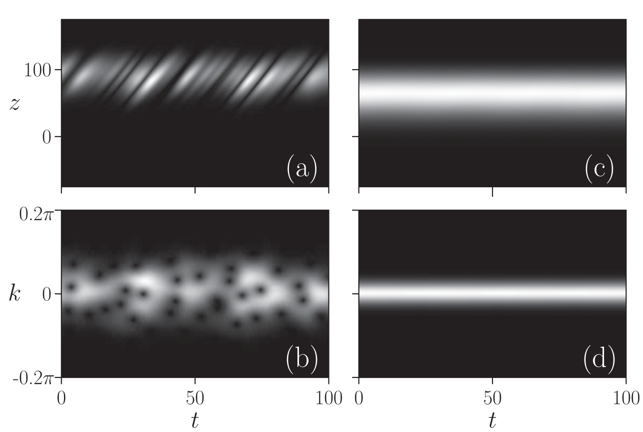


FIG. 4. Numerical result in the global coupling case for a velocity $v = 4$. (a) Without feedback, (c) with delayed feedback ($a = 140$, $\alpha = 10^{-4}$), (b),(d) evolutions of associated spatial Fourier transforms $|\tilde{e}(k,T)|^2$. Other parameters are $R = 1$, $\epsilon = 10^{-2}$, $\sqrt{\eta} = 10^{-10}$.

as is suggested by works on global saturation in different contexts [9]).

In a general way, this type of strategy presents simultaneously strong differences as well as hidden common points with Ott-Grebogi-Yorke (OGY) type methods [10]. When erratic regimes are successfully suppressed, the amplitude of the perturbation applied to the system, $p(t) = \alpha e(z + a, t)$, cannot vanish in the limit of zero noise ($\eta \rightarrow 0$), in contrast to OGY methods. However, in practice, the needed perturbations reach extremely small values [as displayed in Fig. 3(b)]. This was observed in numerical simulations in various situations where (i) transient growth is large, and (ii) the system's size is much larger than the spatial shift. As a reason for this "low-cost," transient growth—though responsible for the turbulent behavior through amplification of noise—also serves as an amplification mechanism for the feedback signal.

In the experimental problem that motivated this work, stabilization of free-electron Laser oscillators (FEL), we will show that the needed feedback is easily implemented by using a delayed optical reinjection. This corresponds to the coherent photon seeding technique (CPS) [2,13,14] which is known to affect the so-called excess noise of mode-locked lasers and synchronously pumped optical parametric oscillators [14].

In these systems, an optical pulse experiences round trips between two mirrors of a cavity, and the evolution of the field pulse shape versus time $T$ is usually described by a master equation with the following structure:

$$e_T(\theta, T) + v e_\theta(\theta, T) = F(e(\theta, T), e_{\theta\theta}(\theta, T), g(\theta, T)), \qquad (8)$$

where $\theta$ is the fast time resolving the pulse shape, and $T$ is the time expressed in units of the field cavity lifetime. $v$ is an adjustable parameter characterizing the mismatch between the cavity round-trip time, and the external forcing frequency. The right-hand side contains diffusion and dispersion terms, a gain term $g$ which usually obeys an additional differential equation [9,15–17], and a small noise source $\xi$ due to spontaneous emission. In the model for our FEL [17], dispersion is, however, negligible, as it operates in a picosecond regime. It has been shown that these lasers possess a dynamics similar to the Ginzburg-Landau Eq. (1) with global saturation coupling. In particular transient growth, through the convective term $v e_\theta$ leads to turbulent behaviors [2,3,9,17], in the sense of noise-sustained structures [1].

CPS consists of a small optical feedback with a delay near the round-trip time $\tau_R$ or one of its multiples $p\tau_R$ [13]. In consequence, CPS can be modeled by the addition of a term $\alpha e(\theta + a, T)$ in Eq. (8), with $\alpha$ the fraction of light reinjected in the mode by the feedback, and $-a$ representing the feedback delay modulo $\tau_R$. In this form, CPS hence appears as a technique for which the present framework holds.

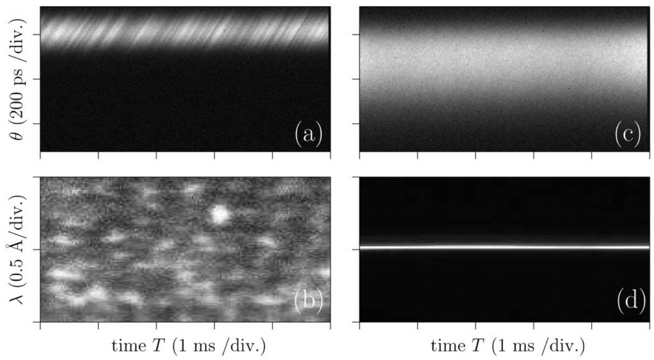


FIG. 5. Experimental feedback-induced erratic regime suppression in a free-electron laser (fraction of reinjected power $\alpha^2 \approx 0.5 \times 10^{-8}$, spatial shift $a = 130$ ps). $\theta$ is the relevant spatial coordinate. Upper and lower figures are $|e(\theta, T)|^2$ and the spectrum versus time $|\tilde{e}(k, T)|^2$, respectively. (a),(b) Without feedback. (c),(d) With feedback. Each streak camera recording is synchronized with its corresponding spectrum.

We tested CPS on the UVSOR-II FEL (Okazaki, Japan) [18] operating at 420 nm, by placing a plane feedback mirror at a distance close to $c\tau_R = 13.3$ m from the output. To our knowledge, no previous results were reported on CPS applied on FELs; thus, we explored systematically the $(\upsilon, a)$ parameter space. Evolutions of pulse profiles $|e(\theta, T)|^2$ and spectra $|\tilde{e}(k, T)|^2$ were recorded in real time using a streak camera (Hamamatsu C5680) and a Fabry-Perot etalon followed by a CCD array. The symmetry $(\upsilon, a) \rightarrow (-\upsilon, -a)$ in Eq. (1) was verified experimentally, and allowed to deduce the mirror position associated to zero delay ($a = 0$). At positive rf frequency detunings (i.e., $\upsilon > 0$ [17]) stabilization systematically required a feedback delay shorter than $\tau_R$ (i.e., $a > 0$) and vice-versa, in agreement with theory. Streak camera recordings [Figs. 5(a) and 5(c)] provided direct evidence of suppression of the noise-sustained structures expected by theory (Fig. 4), and this was clearly correlated to a strong spectral narrowing [Figs. 5(b) and 5(d)] similar to the classical laser case [19]. The optimum value of $a$ typically lead to spectral widths below the spectrometer resolution (0.05 Å FWHM). We calculated the fraction of power reinjected in the $TEM_{00}$ mode of the cavity (from inside to inside) to be as low as $\alpha^2 = 0.5 \times 10^{-8}$.

In conclusion, spatiotemporal systems subjected to advection-induced noise-sustained structures can be stabilized using a simple ''shifted-feedback'' scheme. The process involves the creation of a new steady state solution in the associated deterministic problem. In the case of local coupling, the appearance of the solution suffices to achieve regular behavior, and analytical criteria can be found. In the case of global saturation coupling, the situation is more complicated, as the thresholds for steady state creation and for dropoff of noise sensitivity occur at different parameter values. However, beside these differences, in both cases the minimum perturbations amplitude is limited by the noise level in the system, as in the OGY-type methods. Surprisingly the present stabilization process also gives a framework to the process of coherent photon seeding already used in mode-locked lasers [13].

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