# Distilling Nonlocality

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Two parts of an entangled quantum state can have a correlation, in their joint behavior under measurements, that is unexplainable by shared classical information. Such correlations are called *nonlocal* and have proven to be an interesting resource for information processing. Since nonlocal correlations are more useful if they are stronger, it is natural to ask whether weak nonlocality can be amplified. We give an affirmative answer by presenting the first protocol for distilling nonlocality in the framework of generalized nonsignaling theories. Our protocol works for both quantum and nonquantum correlations. This shows that in many contexts, the extent to which a single instance of a correlation can violate a Clauser-Horne-Shimony-Holt inequality is not a good measure for the usefulness of nonlocality. A more meaningful measure follows from our results.

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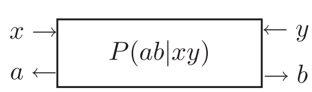
When two separated parts of a quantum state are measured in fixed bases, then the outcomes can show a correlation. Whereas this may be surprising from a physical point of view, it is not from the standpoint of information: such correlations could be explained by randomness shared when the two particles were generated.

If one considers, however, different possible measurement settings on the two sides, then correlations of a stronger kind can arise, which are unexplainable by shared randomness only [1]: This is *nonlocality.*

Quantum mechanics is nonlocal but not maximally so. There are stronger correlations still in accordance with the nonsignaling postulate of relativity [2]. This fact motivated the study of so-called generalized nonsignaling theories [3,4] in which quantum correlations are a special case. Following this general approach to nonlocality, we study correlations between the joint behavior of the two ends of a bipartite input-output *system*, characterized by a conditional probability distribution $P(ab|xy)$. Let $x$ and $a$ be the input and output on the left-hand side of the system, and $y$ and $b$ the corresponding values on the right-hand side.




We call such a system *local* if it is explainable by shared classical information. On the other hand, it is *signaling* if it allows for message transmission in either direction.

John Bell has given certain inequalities that local systems must obey. Hence, *violation* of such an inequality is a witness of nonlocality. In the case where both inputs and both outputs are *binary*, the only such inequality (up to symmetries) is the Clauser-Horne-Shimony-Holt (CHSH) inequality [5]. Furthermore, the set of eight CHSH inequalities is complete for binary systems in the sense that if none of them is violated, then the system is local.

In this Letter we restrict ourselves to the state space of binary input–binary output nonsignaling systems. We refer to [4] for a detailed description of this set.

Nonlocal correlations are not only a fascinating phenomenon, but have as well been shown to be an interesting resource for information processing. Examples include device-independent secrecy of quantum cryptography [6] and nonlocal computation [7]. Furthermore, the existence of nonlocality that is superquantum to some extent would have dramatic consequences on communication complexity [8]. This extends the fact that *maximal* nonlocality would collapse communication complexity, i.e., it allows us to compute every distributed Boolean function with just one communicated bit [9].

The extent by which a Bell, e.g., CHSH, inequality is violated can be taken as a measure for nonlocality. Not surprisingly, nonlocality is a more useful resource, the stronger it is. For instance, the violation of the CHSH inequality gives a lower bound to the uncertainty of a third party about the output bits of a nonsignaling system, which is better the stronger the violation is.

Motivated by these facts, we study the problem of whether nonlocality can be amplified: Can stronger nonlocality be obtained from a number of weakly nonlocal systems? We consider protocols for nonlocality distillation executed by two parties having access to weakly nonlocal systems. The parties on the two sides can carry out arbitrary operations on their pieces of information, but they *cannot* communicate.

Note that such protocols should not be confused with protocols for *entanglement* distillation: There, the input and output are (weakly and strongly, respectively) entangled quantum states, and the allowed operations are classical communication and local quantum operations. The existence of certain entanglement distillation protocols *without communication* is known [10], but this result is independent of ours.

There are several known impossibility results on nonlocality distillation. First, it is not possible to create nonlocality from locality, i.e., to pass the Bell bound [1]. Second, there exists no nonlocality distillation which can pass the Tsirelson bound [11] if the nonlocal systems can be simulated by quantum mechanics. Third, a simple inductive argument shows that a system that exhibits the algebraically maximal possible CHSH violation cannot be obtained from weaker ones. Fourth, it has been shown recently that the CHSH violation of two copies of isotropic systems cannot be distilled [12]. And finally, it has been proven in [13] that there exists an infinite number of isotropic systems for which nonlocality distillation cannot be achieved.

An open question which remains is whether nonlocality can be distilled at all. We answer this question affirmatively.

*Main result.*—There exists a protocol which allows the distillation of certain, both quantum-mechanically achievable and unachievable, binary nonlocal systems.

*Definitions.*—A binary input-output system characterized by a conditional probability distribution $P(ab|xy)$ is *nonsignaling* if one cannot signal from one side to the other by the choice of the input. This means that the marginal probabilities $P(a|x)$ and $P(b|y)$ are independent of $y$ and $x$, respectively, i.e.,

$$\sum_b P(ab|xy) = \sum_b P(ab|xy') \equiv P(a|x) \quad \forall\ a, x, y, y',$$

$$\sum_a P(ab|xy) = \sum_a P(ab|x'y) \equiv P(b|y) \quad \forall\ b, x, x', y.$$

When using a nonsignaling system, a party receives its output immediately after giving its input, independently of whether the other has given its input already. This prevents the parties from signaling by delaying their inputs.

If appropriate we represent a system by its probability distribution $P(ab|xy)$ in matrix notation as

$$\begin{bmatrix} P(00|00) & P(01|00) & P(10|00) & P(11|00) \\ P(00|01) & P(01|01) & P(10|01) & P(11|01) \\ P(00|10) & P(01|10) & P(10|10) & P(11|10) \\ P(00|11) & P(01|11) & P(10|11) & P(11|11) \end{bmatrix}.$$

Given $P(ab|xy)$ $(P)$ we define the set of four correlation functions:

$$X_{xy}(P) = P(00|xy) + P(11|xy) - P(01|xy) - P(10|xy),$$

for $xy = 00, 01, 10, 11$. The corresponding system is local if and only if its correlation functions satisfy the following CHSH inequalities [5]:

$$|X_{xy}(P) + X_{x\bar{y}}(P) + X_{\bar{x}y}(P) - X_{\bar{x}\bar{y}}(P)| \leq 2, \qquad (1)$$

for $xy = 00, 01, 10, 11$. (We use $\bar{x}$ and $\bar{y}$ to indicate bit flips, that is, $\bar{0} = 1$ and $\bar{1} = 0$.)

In order to measure the nonlocality of a system we will use the maximal violation of a CHSH inequality:

*Definition 1.* We define the CHSH nonlocality of a binary input, binary output system $P$ as

$$\mathrm{NL}[P] := \max_{xy} |X_{xy}(P) + X_{x\bar{y}}(P) + X_{\bar{x}y}(P) - X_{\bar{x}\bar{y}}(P)|.$$

Note that $\mathrm{NL}[P] > 2$ indicates that the correlation $P$ violates the CHSH inequality and is therefore called nonlocal.

Quantum mechanics predicts violations of the CHSH inequalities (1) up to $2\sqrt{2}$. However, this bound is only necessary. The necessary *and* sufficient condition for a set of four numbers to be reached by quantum mechanics was found by Landau [14] and Tsirelson [15] (see also Masanes [16]).

*Lemma 1.*—A set of correlation functions $X_{xy}$, $xy = 00$, 01, 10, 11, can be reached by a quantum state and some local observables if and only if they satisfy the following four inequalities:

$$|\arcsin X_{xy} + \arcsin X_{x\bar{y}} + \arcsin X_{\bar{x}y} - \arcsin X_{\bar{x}\bar{y}}| \leq \pi.$$

Using the terms introduced above we formally define a nonlocality-distillation protocol as follows:

*Definition 2.*—A nonlocality-distillation protocol is executed by two parties (Alice and Bob) without communication. It simulates a binary input–binary output system $P^n$ by classical (local) operations on $n$ nonlocal resource systems $P$, such that $\mathrm{NL}[P^n] > \mathrm{NL}[P] > 2$.

*Results.*—In the following we present a nonlocality-distillation protocol and distillable nonlocal resource systems. We will also present resource systems that are measurable on a quantum state and can be used by our protocol to distill (quantum) nonlocality.

We define the nonlocality-distillation protocol $\mathrm{NDP}_n(P)$ on $n$ nonsignaling systems $P$ between Alice and Bob as follows: On inputs $x$ to Alice and $y$ to Bob the parties input $x$ and $y$ to all $n$ systems in parallel and receive outputs $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$, respectively. The parties then locally compute their output bits as $a = \sum_{i=1}^{n} a_i \pmod 2$ for Alice and $b = \sum_{i=1}^{n} b_i \pmod 2$ for Bob. The whole protocol is illustrated in more detail in Fig. 1.

For $0 < \varepsilon \leq 1$ we define the following nonsignaling system

$$P_\varepsilon = \begin{bmatrix} 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 - \varepsilon/2 & \varepsilon/2 & \varepsilon/2 & 1/2 - \varepsilon/2 \end{bmatrix}$$

as our nonlocal distillation resource with CHSH nonlocality $\mathrm{NL}[P_\varepsilon] = 3 - (1 - 2\varepsilon) > 2$. With probability $\varepsilon$ this system violates a CHSH inequality to the algebraic maximum [2] and with probability $1 - \varepsilon$ it outputs perfectly correlated random bits.

*Theorem 1.*—For $n > 1$ and $0 < \varepsilon < 1/2$ the protocol $\mathrm{NDP}_n(P_\varepsilon)$ is a nonlocality-distillation protocol.

*Proof of Theorem 1.*—Obviously, $\mathrm{NDP}_n(P_\varepsilon)$ describes only classical, local operations on Alice's and Bob's side.

| Alice | $\mathrm{NDP}_n(P)$ | BOB |
|---|---|---|
| $x \in \{0,1\}$ | inputs | $y \in \{0,1\}$ |
| $a_1$ | $P(a_1 b_1 \vert xy)$ | $b_1$ |
| $a_2$ | $P(a_2 b_2 \vert xy)$ | $b_2$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $a_n$ | $P(a_n b_n \vert xy)$ | $b_n$ |
| $a = \sum_{i=1}^n a_i \mod 2$ | outputs | $b = \sum_{i=1}^n b_i \mod 2$ |

FIG. 1. The final outputs are a simple exclusive or of all the outputs obtained from a parallel usage of the available nonlocal resource systems.

Furthermore, $\mathrm{NDP}_n(P_\varepsilon)$ simulates another binary input–binary output system $P_\varepsilon^n$ with CHSH nonlocality

$$\begin{aligned}\mathrm{NL}[P_\varepsilon^n] &= \max_{xy} |X_{xy}(P_\varepsilon^n) + X_{x\bar{y}}(P_\varepsilon^n) + X_{\bar{x}y}(P_\varepsilon^n) - X_{\bar{x}\bar{y}}(P_\varepsilon^n)| \\ &= X_{00}(P_\varepsilon^n) + X_{01}(P_\varepsilon^n) + X_{10}(P_\varepsilon^n) - X_{11}(P_\varepsilon^n) \\ &= 3 - X_{11}(P_\varepsilon^n) \\ &= 3 - [P_\varepsilon^n(00|11) + P_\varepsilon^n(11|11) - P_\varepsilon^n(01|11) \\ &\quad - P_\varepsilon^n(10|11)].\end{aligned}$$

Here, we used that $X_{00}(P_\varepsilon^n)$, $X_{01}(P_\varepsilon^n)$, $X_{10}(P_\varepsilon^n)$ are constant functions reaching the algebraic maximum of 1. Analogously to $P_\varepsilon^n$, let $P_\varepsilon^{n-1}$ denote the system simulated by $\mathrm{NDP}_{n-1}(P_\varepsilon)$. Using

$$\begin{aligned}P_\varepsilon^n(00|11) = P_\varepsilon^n(11|11) &= (1/2 - \varepsilon/2)[P_\varepsilon^{n-1}(00|11) + P_\varepsilon^{n-1}(11|11)] \\ &\quad + \varepsilon/2[P_\varepsilon^{n-1}(01|11) + P_\varepsilon^{n-1}(10|11)],\end{aligned}$$

and

$$\begin{aligned}P_\varepsilon^n(01|11) = P_\varepsilon^n(10|11) &= \varepsilon/2[P_\varepsilon^{n-1}(00|11) + P_\varepsilon^{n-1}(11|11)] + (1/2 \\ &\quad - \varepsilon/2)[P_\varepsilon^{n-1}(01|11) + P_\varepsilon^{n-1}(10|11)]\end{aligned}$$

we derive

$$\begin{aligned}\mathrm{NL}[P_\varepsilon^n] &= 3 - (1-2\varepsilon)[P_\varepsilon^{n-1}(00|11) + P_\varepsilon^{n-1}(11|11) \\ &\quad - P_\varepsilon^{n-1}(01|11) - P_\varepsilon^{n-1}(10|11)] \\ &= 3 - (1-2\varepsilon)X_{11}(P_\varepsilon^{n-1}).\end{aligned}$$

Therefore, we have established

$$\begin{aligned}\mathrm{NL}[P_\varepsilon^n] &= 3 - X_{11}(P_\varepsilon^n) = 3 - (1-2\varepsilon)X_{11}(P_\varepsilon^{n-1}) \\ &= 3 - (1-2\varepsilon)^{n-1}X_{11}(P_\varepsilon) \\ &= 3 - (1-2\varepsilon)^n.\end{aligned}$$

For $0 < \varepsilon < 1/2$ we can guarantee $3 - (1-2\varepsilon)^n > 3 - (1-2\varepsilon)^{n-1}$, which implies $\mathrm{NL}[P_\varepsilon^n] > \mathrm{NL}[P_\varepsilon]$. □

In the limit we have $\lim_{n\to\infty}\mathrm{NL}[P_\varepsilon^n] = \lim_{n\to\infty} 3 - (1-2\varepsilon)^n = 3$.

Note that the presented systems are not quantum-physically realizable. This allows our protocol to pass the Tsirelson bound using $P_\varepsilon$ with $0 < \varepsilon \le \sqrt{2} - 1$ as resource systems. In the following we show that nonlocality distillation is also possible for systems available in quantum mechanics. We therefore introduce a more general parameterized system (positivity is ensured by $0 \le \varepsilon, \delta \le 1$):

$$P_{\varepsilon,\delta} = \begin{bmatrix} 1/2 - \delta/2 & \delta/2 & \delta/2 & 1/2 - \delta/2 \\ 1/2 - \delta/2 & \delta/2 & \delta/2 & 1/2 - \delta/2 \\ 1/2 - \delta/2 & \delta/2 & \delta/2 & 1/2 - \delta/2 \\ 1/2 - \varepsilon/2 & \varepsilon/2 & \varepsilon/2 & 1/2 - \varepsilon/2 \end{bmatrix}.$$

This system has CHSH nonlocality $3(1-2\delta) - (1-2\varepsilon)$. For $\delta = 0$ we have $P_{\varepsilon,\delta} = P_\varepsilon$.

Note that we have chosen the two example resource systems because of their simplicity. This should not suggest that these exact systems are the only systems distillable by our protocol. Obviously the distillability of a system with the presented protocol does only depend on its correlation functions and not on the marginals.

*Theorem 2.*—There exist $0 < \delta < \varepsilon < 1/2$ and $n > 1$ such that $P_{\varepsilon,\delta}$ is a quantum system and $\mathrm{NDP}_n(P_{\varepsilon,\delta})$ is a nonlocality-distillation protocol.

*Proof of Theorem 2.*—Protocol $\mathrm{NDP}_n(P_{\varepsilon,\delta})$ simulates another two input/two output system $P_{\varepsilon,\delta}^n$. By setting $\delta < \varepsilon$ and following a similar reasoning as in the proof of Theorem 1 we obtain

$$\begin{aligned}\mathrm{NL}[P_{\varepsilon,\delta}^n] &= X_{00}(P_{\varepsilon,\delta}^n) + X_{01}(P_{\varepsilon,\delta}^n) + X_{10}(P_{\varepsilon,\delta}^n) \\ &\quad - X_{11}(P_{\varepsilon,\delta}^n) \\ &= 3(1-2\delta)^n - (1-2\varepsilon)^n.\end{aligned}$$

We can find values $n$ and $0 < \delta < \varepsilon < 1/2$ (for example, $n = 2$, $\varepsilon = 0.01$, $\delta = 0.002$) such that $P_{\varepsilon,\delta}$ is at the same time distillable, i.e.,

$$3(1-2\delta)^n - (1-2\varepsilon)^n > 3(1-2\delta) - (1-2\varepsilon)$$

and a quantum system, i.e.,

$$|3\arcsin(1-2\delta) - \arcsin(1-2\varepsilon)| \le \pi,$$
$$|\arcsin(1-2\delta) + \arcsin(1-2\varepsilon)| \le \pi.$$

Lemma 1 only guarantees that the correlation functions of $P_{\varepsilon,\delta}$ are obtainable by quantum mechanics. But Alice and Bob can make their outputs locally uniform such that the correlation functions are preserved using shared randomness. Thus, $P_{\varepsilon,\delta}$ is a quantum system if its correlation functions are obtainable by quantum mechanics.

Therefore, we can achieve $\mathrm{NL}[P^n_{\varepsilon,\delta}] > \mathrm{NL}[P_{\varepsilon,\delta}]$, which means that nonlocality has been distilled with quantum systems as resources. □

A natural follow up question concerns the *maximum* nonlocality our protocol can distill using the quantum systems presented above.

Optimal parameters $n$, $\varepsilon$, $\delta$ maximize the term $\mathrm{NL}[P^n_{\varepsilon,\delta}] = 3(1-2\delta)^n - (1-2\varepsilon)^n$ with respect to the conditions that $\mathrm{NL}[P^n_{\varepsilon,\delta}] > \mathrm{NL}[P_{\varepsilon,\delta}]$ and that $P_{\varepsilon,\delta}$ is a quantum system (Lemma 1). The maximal nonlocality that can be distilled by $\mathrm{NDP}_n(P_{\varepsilon,\delta})$ is

$$\mathrm{NL}[P^{n_{\max}}_{\varepsilon_{\max},\delta_{\max}}] = 1 + \sqrt{2},$$

where $n_{\max} = 2$, $\varepsilon_{\max} \simeq 0.30866$ and $\delta_{\max} \simeq 0.03806$.

*A new measure of nonlocality.*—The possibility of distillation motivates the definition of a new measure for nonlocality, namely, the maximal CHSH violation achievable from many realizations of a given system by any distillation protocol.

As an example application consider the computation of the nonlocally distributed version of the AND function: Two separated parties are given inputs $x_1$, $x_2$ and $y_1$, $y_2$, respectively, and have to find outputs $a$ and $b$, such that the probability of obtaining

$$a \oplus b = (x_1 \oplus y_1) \wedge (x_2 \oplus y_2) \tag{2}$$

is maximal. Quantum mechanics allows no advantage over the optimal, classical strategy [7]. Rearranging (2) yields a strategy with success probability directly related to the CHSH violation of a given resource system. By nonlocality distillation of copies of our arbitrarily weak nonlocal system $P_\varepsilon$ a higher success probability above the quantum bound can be reached. This illustrates that distillable systems like $P_\varepsilon$—although located arbitrarily "close" to the quantum bound—are a stronger computational resource than any quantum system. Therefore, we obtain a separation of quantum and post-quantum correlations below the Tsirelson bound in terms of information processing power.

*Conclusion.*—We have shown that nonlocality of binary input–binary output systems, measured by how strongly the CHSH inequality is violated, can be amplified. More precisely, we have shown that certain systems which exhibit an arbitrarily weak violation of a CHSH inequality (achieving $2 + 2\varepsilon$), but that are nevertheless *not* realizable by quantum physics, can be distilled.

Furthermore, we show that even certain quantum-mechanically achievable systems can be distilled: Interestingly, the achievable limit by our protocol is then the exact mean $(1+\sqrt{2})$ between the classical (2) and the quantum $(2\sqrt{2})$ bounds.

Our result complements previous ones, stating that the distillability of nonlocality of two *isotropic* systems is impossible [12] and at most very limited in general [13]. Isotropic systems are an important special case because they are the worst case with respect to distillability, i.e., every nonsignaling system can be turned into an isotropic system such that nonlocality is preserved using shared randomness only (this transformation is known as *depolarization* [17]). Therefore, these nondistillable isotropic systems cannot be used to simulate the distillable resources defined here. In other words, bipartite isotropic and non-isotropic nonsignaling (and quantum) systems are in general inequivalent correlations, although they exhibit the same violation of the CHSH inequality.

The possibility of distillation motivates the definition of a new measure of nonlocality. Clearly, this measure is significant in any context where nonlocality is used as a resource for information processing, and where the number of realizations available is not limited to one.

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