Fractal Properties of Quantum Spacetime

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We show that, in general, a spacetime having a quantum group symmetry has also a scale-dependent fractal dimension which deviates from its classical value at short scales, a phenomenon that resembles what is observed in some approaches to quantum gravity. In particular, we analyze the cases of a quantum sphere and of κ-Minkowski spacetime, the latter being relevant in the context of quantum gravity.

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In the quest for new physics at the Planck scale, the idea that spacetime might become noncommutative has gained a lot of attention, in particular, for its potential phenomenological implications [1]. Whether such an idea is supposed to be taken as a starting point for the construction of a quantum theory of gravity (see, for example, [2]) or can be derived from it (for example, [3]), there are several reasons why it could play a role at the Planck scale. Somehow postponing the issue of a more complete and fundamental theory, most of the effort in the literature has gone into the study of noncommutative versions of flat spacetime, which naively might be thought of as a ground state of the full theory of quantum gravity.

On the other hand, constructive approaches to quantum gravity, such as causal dynamical triangulations (CDT) [4] and the exact renormalization group (ERG) [5], which make no use of postulated new physics, have something interesting to say about Planck scale properties of spacetime. It is somehow surprising to see that apparently very different approaches give rise to very similar results, as is the case for the spectral dimension of spacetime: Both in CDT [6] and in ERG [7] evidence has been given for the emergence of a (ground state) spacetime with fractal properties such as the effective (spectral) dimension d_s varying from a classical value $d_s = 4$ at large scales down to $d_s = 2$ at short scales, a result which is suggestive of how gravity might cure its own ultraviolet behavior via a dynamical dimensional reduction. It is a legitimate and interesting question to ask whether such a fractal nature of spacetime at short scales is compatible with the expectation of some sort of noncommutativity.

An appealing realization of noncommutativity is that in which spacetime remains maximally symmetric but the Lie group of symmetries is deformed into a quantum group (as in [8]), a deformation also favored by general arguments on the possible nonlocality of a final quantum theory of gravity [9] and which constitutes a solid realization of the so-called doubly special relativity [10,11]. Research in this area is still at an early stage, and a complete formulation of quantum field theory based on a quantum group symmetry is still lacking, but some proposals have been put forward for the construction of the corresponding Fock space (see, for example, [12] and references therein). Here we consider two particular examples, the quantum sphere and κ-Minkowski spacetime, and, by calculating their spectral dimension, we infer some general properties for such types of spaces. In order to do so, we adopt a group theoretical construction that suits well the quantum group formalism. For the noncommutative spacetimes considered, we find a result qualitatively similar to that found in CDT and ERG, i.e., a scale-dependent spectral dimension which reaches its classical value only at large scales.

Spectral dimension.—A possible way to study the geometry of a Riemannian manifold M with metric $g_{\mu\nu}$ is via the spectral theory of the scalar Laplacian $\Delta = -g^{\mu\nu}\nabla_\mu\nabla_\nu$, where ∇_μ is the covariant derivative. To such an operator can be associated a heat kernel, i.e., a function $K(x, y; s)$ on $M \times M \times \mathbb{R}_+$ which solves the heat equation

$$\partial_s K(x, y; s) + \Delta_x K(x, y; s) = 0, \tag{1}$$

with the initial condition $K(x, y; 0_+) = \delta(x - y)/\sqrt{g(x)}$.

It is a well known result that the (normalized) trace of the heat kernel has the expansion

$$\mathrm{Tr}\, K = \frac{\int d^n x \sqrt{g(x)} K(x, x; s)}{\int d^n x \sqrt{g(x)}} \sim \frac{1}{(4\pi s)^{n/2}} \sum_{i=0}^{+\infty} a_i s^i, \tag{2}$$

where the coefficients are metric-dependent invariants which can be calculated via recursion formulas, with $a_0 = 1$. It is then possible to define the notion of *spectral dimension* by the formula

$$d_s \equiv -2 \frac{\partial \log \mathrm{Tr} K}{\partial \log s}. \tag{3}$$

On flat space $a_i \equiv 0$ for $i \geq 1$, and so we recover $d_s = n$. On a general classical curved space, $d_s = n$ only at small s, while deviations occur at large s due to the curvature. Since we can identify the diffusion time s with the scale at which we probe the manifold, when applying the classical expansion (2) to our spacetime, we should take s to be small compared to the characteristic dimension of the space but still large compared to the Planck scale, or else this formula will not be valid anymore because of the metric fluctuations, as suggested by the results in Refs. [6,7].

The usefulness of such a definition is in providing an operational notion of dimension, which is a valuable alternative to the maybe more famous Hausdorff dimension associated to the scaling exponent of the volume of a ball.

The solution of (1) is given by $K = \langle x|e^{-s\Delta}|y\rangle$ or in terms of eigenvalues λ_j and eigenfunctions $\phi_j(x)$ of Δ:

$$K(x, y; s) = \sum_j e^{-\lambda_j s}\phi_j(x)\phi_j^*(y), \tag{4}$$

where it has to be understood that the spectrum might be a continuum and in such a case the sum would be replaced by an integral. In flat spacetime, for example, we have

$$K_{\text{flat}}(x, y; s) = \int \frac{d^n p}{(2\pi)^n} e^{-p^2 s} e^{ip(x-y)} = \frac{e^{-(|x-y|^2/4s)}}{(4\pi s)^{n/2}}. \tag{5}$$

We now want to generalize this notion to a noncommutative space of the kind associated to a quantum group symmetry. In such a space, it is natural to define the Laplacian from the quadratic Casimir operator of the quantum group, in analogy to the general construction on homogeneous spaces [13]. For example, in the case of a flat Euclidean space $E^n \sim ISO(n)/SO(n)$, we find that the spectrum of the Laplacian is given by the first Casimir operator $\mathcal{C}_1 = P_\mu P^\mu$ in the irreducible scalar representations, thus recovering (5). We can follow this route in a straightforward way for the case of a quantum group and, in particular, for a quantum deformation of the Poincaré group, as we will now show.

A toy example: The sphere vs the quantum sphere.—To illustrate the idea, it is useful to look at a simple example first. Following Ref. [13], we write the heat kernel on a homogeneous space as an integral of the heat kernel on the symmetry group G over the isotropy group H:

$$K_{G/H}(x, y; s) = \int_H K_G(e, gh; s)dh, \tag{6}$$

where $y = gx$, $g \in G$, and K_G can be obtained by a character expansion

$$K_G(g; s) \equiv K_G(e, g; s) = \frac{1}{V_G}\sum_j d_j \chi_j(g) e^{-s\mathcal{C}(j)}, \tag{7}$$

where the sum is over all of the irreducible representations of G, d_j is their dimension, $\chi_j(g)$ is the character of $g \in G$ in the representation j, $\mathcal{C}(j)$ is the value of the Casimir operator in that representation, and V_G is the volume of G. Plugging (7) into (6), one finds that the integration restricts the summation to be only over the spherical representations of G with respect to H, i.e., those which contain the singlet of H.

Our first example, before moving to quantum spaces, is the classical (unit) two-sphere S^2 considered as $SU(2)/U(1)$. Using (7) and (6), one finds the expression

$$K_{S^2}(\theta; s) = \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l + 1)P_l(\cos\theta)e^{-sl(l+1)}, \tag{8}$$

where $P_l(x)$ are the Legendre polynomials and $\mathcal{C}(l) = l(l + 1)$ and which is of course equivalent to the expression (4). Taking the trace is trivial, and we find

$$\text{Tr}\, K_{S^2} = \sum_{l=0}^{\infty}(2l + 1)e^{-sl(l+1)}. \tag{9}$$

Finally, we can use formula (3) to get the spectral dimension, which we plot in Fig. 1(a). Note that the value $d_s = 2$ is reached exactly at $s = 0$, and away from that it decreases due to the curvature.

Consider now replacing the group $SU(2)$ by the quantum group $SU_q(2)$ for real q, which is generated by the operators J_+, J_-, and J_3 obeying the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = \frac{\sinh(zJ_3)}{\sinh(z/2)}, \tag{10}$$

where $z = \ln q$. Such generators belong to the quantum Hopf algebra $U_q[su(2)]$ whose representations are well known (see, for example, [14]) and parallel (for real q) those of $su(2)$, in the sense that for every $j = 0, \frac{1}{2}, 1, \ldots$ the Hopf algebra $U_q[su(2)]$ has a $(2j + 1)$-dimensional representation $\{|j, m\rangle, m = -j, -j + 1, \ldots, j\}$, with $J_3|j, m\rangle = m|j, m\rangle$ [the action of $J_\pm$ and the coalgebra structure are different from those of $su(2)$ but we do not need them here]. The Casimir operator in the representation j is given by

$$\mathcal{C}(j) = \frac{\cosh[z(2j + 1)/2] - \cosh(z/2)}{2\sinh^2(z/2)}. \tag{11}$$

The above steps for the case of $SU(2)$ can be repeated for $SU_q(2)$; in particular, the integration over $U(1)$ restricts the sum over representations to only those with integer j, i.e., those containing the singlet of $U(1)$ which correspond to

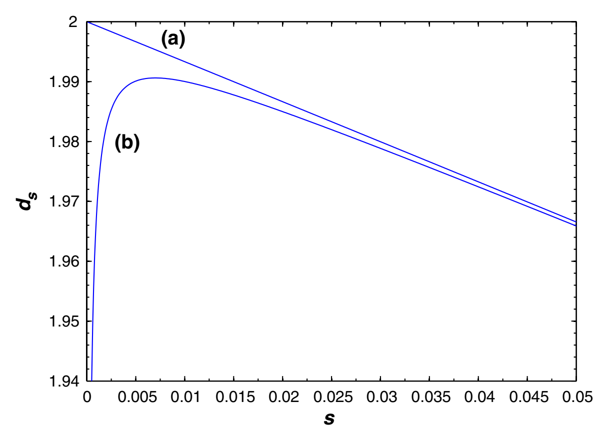

FIG. 1 (color online). (a) The spectral dimension of a unit sphere $SU(2)/U(1)$. (b) The case of a quantum sphere $SU_q(2)/U(1)$, for $z = 0.01$.

$m = 0$. We only need to replace in (9) the standard $su(2)$ Casimir operator with (11).

Again the spectral dimension can be computed (numerically) using (3), and the result is plotted in Fig. 1(b). Clearly, the behavior is the standard one for large s, but it deviates sensibly as s decreases, with d_s never reaching the value $d = 2$ and going instead down to zero. We can think of this phenomenon as a signature of the fuzziness of the quantum sphere or of fractal behavior at short scales.

κ-Minkowski spacetime.—κ-Poincaré algebra was derived in Ref. [15] as a particular contraction of the quantum anti–de Sitter algebra $U_q[O(3,2)]$ in which the anti–de Sitter radius R goes to infinity while $R\ln q = \kappa^{-1}$ is a real number which is held fixed and finite, a limit which might be of relevance for a theory of quantum gravity [16]. The result of such contraction is another Hopf algebra which is conveniently expressed in terms of the so-called (symmetric) bicrossproduct basis [8], such that the bicrossproduct structure of κ-Poincaré algebra $\mathcal{P}_\kappa = U[so(3,1)] \triangleright\!\blacktriangleleft T$ becomes evident, with generators of rotation M_j and boost N_j forming the standard Lorentz algebra and with deformed action of $U[so(3,1)]$ on the translation sector T given by the remaining commutators

$$[N_i, P_0] = e^{-(P_0/2\kappa)} P_i,$$
$$[N_i, P_j] = \delta_{ij} e^{-(P_0/2\kappa)}\left(\kappa \sinh\frac{P_0}{\kappa} + \frac{1}{2\kappa}\vec{P}^2\right) - \frac{1}{2\kappa} e^{-(P_0/2\kappa)} P_i P_j. \tag{12}$$

As shown in Ref. [17], Hermitian irreducible representations of the Poincaré algebra with $\mathcal{C}_1 = P_\mu P^\mu \geq 0$ can be lifted to Hermitian irreducible representations of κ-Poincaré algebra with $\mathcal{C}_1^\kappa = (2\kappa \sinh\frac{P_0}{2\kappa})^2 - \vec{P}^2 \geq 0$, and the latter reduce in the $\kappa \to \infty$ limit to the undeformed ones.

κ-Minkowski spacetime was introduced in Ref. [8] as the space which is dual to the translation sector of κ-Poincaré algebra and on which the whole κ-Poincaré algebra acts covariantly and, as such, is a subgroup of the so-called κ-Poincaré group [18]. It turns out to be a noncommutative spacetime with coordinates $\hat{x}^\mu$ satisfying the relations

$$[\hat{x}^0, \hat{x}^j] = \frac{i}{\kappa}\hat{x}^j, \qquad [\hat{x}^i, \hat{x}^j] = 0. \tag{13}$$

One nice consequence of the bicrossproduct structure is that the dual P_κ^* possesses the same structure, i.e., $P_\kappa^* = T^* \blacktriangleright\!\triangleleft \mathbb{C}[SO(3,1)]$, and so we can think of κ-Minkowski spacetime as the homogeneous space $P_\kappa^*/SO(3,1)$, and this justifies us in applying the previous formalism for evaluating the trace of the heat kernel on κ-Minkowski spacetime.

Before doing that, we have to switch to the Euclidean signature in order to make sense of our definition of effective dimension, but this constitutes no problem; it just amounts to the substitution (see, for example, [19]) $P_0 \to iP_0$, $\kappa \to i\kappa$. When applied to the first Casimir operator of the algebra, such a substitution yields

$$\mathcal{C}_1^\kappa = \left(2\kappa \sinh\frac{P_0}{2\kappa}\right)^2 + \vec{P}^2, \tag{14}$$

in agreement with a naive extension of the two- and three-dimensional cases [20].

Next, we also have to note that any function of $\mathcal{C}_1^\kappa$ is still a valid Casimir operator, because by having an extra parameter which is dimensionful we can construct arbitrary functions with mass-squared dimension, the only restriction being given by the classical limit $\kappa \to \infty$. To select one unique expression, we can make an appeal to the existing theory of differential calculus on κ-Minkowski spacetime [21] (see also [22] for recent applications to quantum field theory on κ-Minkowski spacetime) and compare our group theoretical construction with the Laplacian defined via such differential calculus. We find that, in the basis we have chosen, the eigenvalues of the Laplacian are given by

$$M^2(p) = \mathcal{C}_1^\kappa(p)\left(1 + \frac{\mathcal{C}_1^\kappa(p)}{4\kappa^2}\right). \tag{15}$$

We can now use $M^2(p)$ as the Casimir eigenvalue and write down the following formula for κ-Minkowski spacetime:

$$\mathrm{Tr}\, K_q = \int \frac{d\mu(p)}{(2\pi)^n} e^{-sM^2(p)}, \tag{16}$$

where we have also used the κ-deformed Lorentz invariant measure $d\mu(p) \equiv e^{(3p_0/2\kappa)} d^4p$. Finally, from (3) we obtain the spectral dimension of (the Wick-rotated) κ-Minkowski space. The integration cannot be done analytically, but numerically it poses no problems, and we can plot the result as, for example, in Fig. 2. The limiting values at $s \to \infty$ and $s \to 0$ can be obtained analytically by taking the limits of, respectively, small and large p_0/κ for the integrand, obtaining

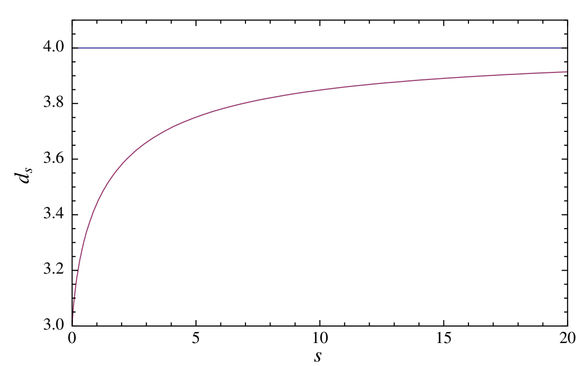

FIG. 2 (color online). A plot of the spectral dimension d_s of κ-Minkowski space for $\kappa = 1$ as a function of the diffusion time s. For comparison, we plot also the constant behavior of the spectral dimension of classical Minkowski space ($d_s = 4$).

$$d_s = \begin{cases} 4 & \text{for } s \to \infty, \\ 3 & \text{for } s \to 0. \end{cases} \tag{17}$$

The behavior in (17), our main result, is qualitatively similar to those in Refs. [6,7] with the main difference being the short scale behavior leading to an effective dimension $d_s = 3$ in our case rather than $d_s = 2$ as in Refs. [6,7]. The fact that the effective dimensionality departs from an integer value, and from the topological dimension, in particular, is a typical signature of fractal geometry. The meaning of fractal in the context of noncommutative geometry is actually a largely unexplored subject, but certainly the behavior found for the spectral dimension can be interpreted qualitatively as a defining property of fractal nature.

The result (17) can also be understood by noticing that the dispersion relation (14) looks like that associated to a finite difference operator (along the time direction). Such an interpretation of the κ-deformed Klein-Gordon operator was known since the early days of κ-Poincaré algebra [23]. In light of this analogy, one might then think of the diffusion process being trapped at short diffusion times s in a continuous three-dimensional slice and that only at large scales the discreteness in time t would become irrelevant and thus look like an additional continuous dimension. On the other hand, the analogy is purely formal as one should notice that the finite difference operator acts along the imaginary axis and that there is actually no discretization of time in the κ-Minkowski construction (t can take any value). For these reasons, we prefer to think of (17) as a result of the noncommutativity of spacetime at the short scale, with the consequent uncertainty relations that would allow one to precisely determine three space coordinates but not the fourth (time).

Conclusions.—We have shown how the result of Refs. [6,7] about the dynamical dimensional reduction at short scales can be reproduced, in its qualitative aspect, by a noncommutative spacetime with quantum group symmetry. In light of the comment above, it is tempting to conjecture that the value of the spectral dimension in the far ultraviolet limit is generally given by the dimension of the maximal commutative subspace. If that turns out to be true, at least within a certain hypothesis (for the quantum sphere above, it is not true), then it would be easy to construct a spacetime whose spectral dimension goes to 2 in the UV, thus paralleling the result in Refs. [6,7] also quantitatively. Less trivial is to identify the associated quantum group and thus prove such a conjecture. We hope to come back to this issue in the near future.

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