Quantum Communication beyond the Localization Length in Disordered Spin Chains

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We study the effects of localization on quantum state transfer in spin chains. We show how to use quantum error correction and multiple parallel spin chains to send a qubit with high fidelity over arbitrary distances, in particular, distances much greater than the localization length of the chain.

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The reliable communication of quantum states from one location to another is most likely necessary if large-scale quantum computing is ever to be realized. Several years ago, Bose [1] proposed using spin chains as the medium for such quantum state transfer. Since then, there has been much interest in this area, and numerous protocols have been put forward which develop this idea [2–9]. Unfortunately, any real spin chain will inevitably have an element of disorder inherent in the system. As pointed out in [10], this can cause a phenomenon known as Anderson localization [11] to take place. This is the process in which the energy eigenstates of a disordered lattice become localized in space, rather than extending throughout the system as they would in the absence of disorder. This in turn inhibits state transfer beyond a distance known as the localization length of the chain and provides a challenge for the use of spin chains in quantum communication. Although localization has been well studied in solid state physics, its implications for quantum information are not well understood [10,12–15]. Indeed, it is not clear at first glance how the problems due to localization fit within the standard error paradigm considered in quantum information theory. First, rather than being due to some coupling of the spin chain with the environment, localization is an intrinsic source of error. Even in the absence of disorder, the excitations carrying the quantum information become spread out along the chain. In the presence of disorder, only an exponentially small part of the signal reaches beyond the localization length. Also, although we are only trying to communicate a single qubit, the localization errors take place in the larger Hilbert space of the entire chain.

In this Letter, we look more closely at the effect localization has on spin chain state transfer and find that, for a class of standard spin chain protocols, localization can effectively be viewed as a source of amplitude damping errors, where the damping parameter is dependent on the distance propagated and the degree of disorder in the chain. We then show how to use multiple spin chains and quantum error correction [16,17] to achieve high fidelity quantum information transfer over arbitrary distances, in particular, over distances much greater than the localization length. By considering a concatenation scheme, we show that if the disorder is not too great, the number of spin chains required scales only polylogarithmically with the distance over which we wish to communicate.

A spin in the $|1\rangle$ state will be called an *excitation*. For a system of N spins, we shall use a bold font $|\mathbf{0}\rangle = |00\ldots0\rangle$ to denote the zero excitation state, and $|\mathbf{j}\rangle = |0\ldots010\ldots0\rangle$ to denote the single excitation state with the jth spin in the state $|1\rangle$, and all others in state $|0\rangle$.

Let us consider the following communication scenario. Alice and Bob are at opposite ends of a chain of N spin-1/2 particles described by some nearest-neighbor Hamiltonian H_0. Alice's task is to send, with as high a fidelity as possible, an unknown qubit state to Bob. We shall assume that (1) The system is isolated from the environment, and thus there are no external sources of noise. (2) H_0 commutes with the total Z-spin operator $\sum_k^N \sigma_k^z$ and hence conserves the number of excitations on the chain. (3) The system starts in the initial state $|\mathbf{0}\rangle$. For example, H_0 could be a simple Heisenberg coupling (in the absence of an external field)

$$H_0 = -(J/2)\sum_i \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} \tag{1}$$

where J is the coupling constant between spins. Note that we have not yet introduced disorder into the system.

The communication proceeds according to [2,3]. We assume that Alice and Bob each have access to a number Λ of spins at their ends of the chain. To begin, Alice encodes the input state $a|0\rangle + b|1\rangle$ as a state of her Λ spins in such a way that $|0\rangle$ is encoded as $|\mathbf{0}\rangle_A$, and $|1\rangle$ is encoded as $|1_{\mathrm{ENC}}\rangle_A$. Here, A denotes Alice's addressable spins, and $|1_{\mathrm{ENC}}\rangle_A$ is a superposition of single excitation states, where the excitations lie in Alice's domain. The system then undergoes unitary evolution for some time t. Note that as a result of Alice's encoding, and the fact that the Hamiltonian preserves the total number of excitations, the chain dynamics remain restricted to the $N+1$ dimensional subspace spanned by the zero and single excitation states. Finally, Bob applies a decoding unitary to his addressable spins. This concentrates the state onto a single spin which he then takes as the output of the transfer. The whole process is equivalent to sending Alice's original state down an amplitude damping channel with time-dependent channel parameter $\gamma(t) = 1 - \mathcal{C}_B(t)$, where

$0 \leq C_B(t) \leq 1$. The corresponding average fidelity is given by $1/2 + \sqrt{C_B(t)}/3 + C_B(t)/6$.

There is a convenient way to visualize the transfer process [2]. At time t, the state of the system can be expressed as $a|\mathbf{0}\rangle + b\sum_{j=1}^{N} c_j(t)|\mathbf{0}\rangle$, where $\sum_{j=1}^{N} |c_j(t)|^2 = 1$. By plotting the quantities $|c_j(t)|^2$ against site number j, we can produce a graph of the state. Then, the quantity $\mathcal{C}_B(t)$ is given by the area under the graph supported by Bob's accessible sites. To achieve high fidelity transfer, the strategy is to choose $|1_{\mathrm{ENC}}\rangle_A$ to be a wave packet with a particular shape that leads to minimal dispersion.

What happens now if there is disorder present in the chain? Any real chain will, due to engineering limitations and thermal fluctuations, have spin-spin couplings that are not described by an ideal Hamiltonian such as (1). In general, there may be complicated, time-dependent perturbations to the Hamiltonian. However, here we consider the simpler case of quenched disorder, with the additional requirement that the perturbations do not couple the zero and single excitation subspaces with each other, nor with subspaces of larger numbers of excitations. For example, the total Hamiltonian might take one of the following forms, corresponding to diagonal and off-diagonal disorder, respectively,

$$H^{\epsilon}_{\mathrm{diag}} = H_0 + \sum_j \epsilon_j |\mathbf{j}\rangle\langle\mathbf{j}| \tag{2}$$

$$H^{\Delta}_{\mathrm{off\text{-}diag}} = H_0 + \sum_j \Delta_j (|\mathbf{j}\rangle\langle\mathbf{j}+\mathbf{1}| + |\mathbf{j}+\mathbf{1}\rangle\langle\mathbf{j}|) \tag{3}$$

where H_0 is the Hamiltonian (1) and the ϵ_j and Δ_j are i.i.d real random variables drawn from some distribution $\mathcal{P}_\delta$ with bounded density, characterized by a disorder parameter δ (e.g., the uniform distribution in the interval $[-\delta, \delta]$). It will become clear, however, that our results do not depend on the details of the type of disorder present.

What are the implications of this for state transfer? In spite of the disorder, the all zero state $|\mathbf{0}\rangle$ trivially remains an energy eigenstate and, furthermore, the chain evolution remains restricted to the zero and single excitation subspaces. This implies that the disordered chain still behaves like an amplitude damping channel, where the damping parameter depends on the area under the graph in Bob's domain. Figure 1 shows how increasing the degree of diagonal disorder causes the graph of the state to suffer from dispersion and reflection as it propagates. Consequently, the area under the graph in Bob's domain, and hence the transfer fidelity, becomes increasingly suppressed. With a disordered chain, the channel parameter $\mathcal{C}_{B,\delta}(t)$ depends on both the time and the particular realization of the disorder [that is, the specific values of the ϵ_j and Δ_j in (2) and (3), respectively]. However, for a given $\mathcal{P}_\delta$, the specific values of ϵ_j (resp. Δ_j) are probabilistic. Thus, $\gamma \equiv \gamma_\delta(t) = 1 - \mathcal{C}_{B,\delta}(t)$ is a stochastic function of time, parameterized by δ.

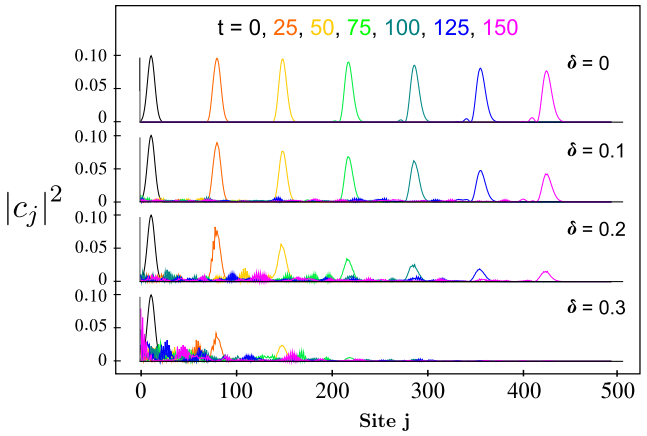

FIG. 1 (color online). Effect of localization on state propagation for a disordered Heisenberg chain with $N = 501$, $J = 1/\sqrt{2}$. The wave packets have been plotted at time increments $\delta t = 25$. Diagonal disorder was drawn from a normal distribution with mean zero and standard deviation δ.

We have investigated this claim in more depth by numerically evaluating $\bar{\gamma}_\delta(t)$ for various values of t and δ, for the cases of both diagonal and off-diagonal disorder, and for various disorder distributions $\mathcal{P}_\delta$. Since the outcome is stochastic, we average over many trials in order to build up a mean surface. This was found to have the empirical form

$$\bar{\gamma}_\delta(t) \sim 1 - e^{-\alpha t(\delta^2 + \beta\delta)} \tag{4}$$

for some constants α and β. Figure 2 shows the results for diagonal disorder, with $\mathcal{P}_\delta$ chosen to be the uniform distribution. The same empirical form (4) was found to hold for off-diagonal disorder, and for $\mathcal{P}_\delta$ chosen to be Cauchy and normal distributions, although of course the quantitative details (the values of α and β) differed. Thus,

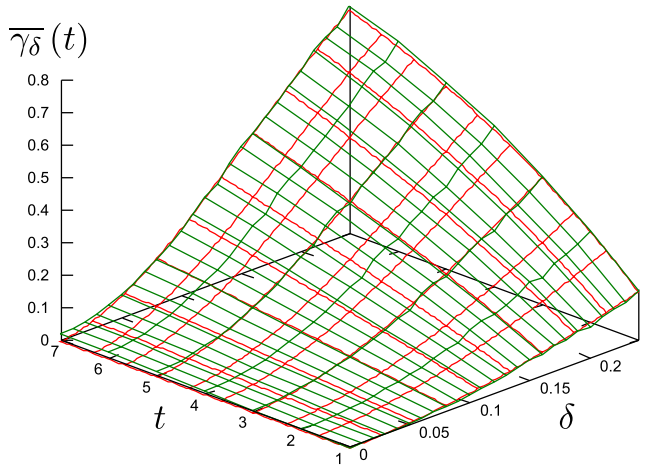

FIG. 2 (color online). Mean surface of $\gamma(t, \delta)$. Green: numerical data. Red: $1 - e^{-\alpha t(\delta^2+\beta\delta)}$, for $\alpha = 2.56$, $\beta = 0.029$. Diagonal disorder drawn from a uniform distribution in the range $[-\delta, \delta]$. Similar results also hold for off-diagonal disorder.

if we know the value of δ, we can deduce how far Bob can be from Alice before, on average, the fidelity drops below a certain threshold. As expected, for fixed δ the fidelity decays exponentially quickly as the state propagates along the chain.

The identification of a disordered spin chain as an amplitude damping channel (albeit one with a stochastic damping parameter) immediately opens up the possibility of using quantum error correction to improve the channel fidelity. Alice can encode her message state into a state of multiple qubits and send each of these qubits down a separate, parallel, spin chain. Bob will then receive the multiple qubits and apply standard error correction techniques. Indeed, quantum codes specifically tailored to treating amplitude damping errors are already known [18]. However, if Alice and Bob are separated by a distance larger than the localization length, the amplitude damping will become too severe, and this procedure will fail. However, we have the option of performing error correction at regular intervals (shorter than the localization length) along the chain, intervening before the amplitude damping becomes too great. Thus, as long as δ is not too large, error correction can be used to correct localization errors and achieve high fidelity transfer over distances much larger than the localization length. Furthermore, as the wave packets propagate with a well-defined group velocity, the transmission takes a time linear in the distance over which we want to propagate.

The key result of this Letter is that this scheme is scalable. That is, the number of parallel chains needed to faithfully communicate a qubit grows favorably with the distance we wish to send it. The proof of this is based on the following protocol. (1) Alice encodes her initial qubit in the space of $n = 5^k$ qubits according to the 5 qubit code [19] concatenated k times. (2) Each of these qubits is then further encoded by Alice as a minimally dispersive wave packet, and sent down a separate spin chain—modified via a channel twirling process [20]—towards Bob. (3) At periodic intervals of distance L, the (distorted) wave packets are decoded down to the space of 5^k qubits, and error correction is applied on these qubits. They are then reencoded back as wave packets to be sent another distance L down the chain. This process is repeated (with twirling applied to each section of length L of the chain) until the wave packets reach Bob's end of the chain, a distance mL away, where m is an integer. (4) When the wave packets reach Bob, he decodes them down to the space of 5^k qubits, and then further decodes these down to the space of a single qubit, which he takes as the output of the channel.

Provided that the number of parallel chains n scales polylogarithmically in the distance we wish to communicate over, it is possible to send a qubit an arbitrary distance, with arbitrarily high fidelity using this protocol. Let us look at the protocol in more detail. We have seen that if Alice sends a qubit, encoded as a wave packet, down a disordered spin chain of length L to Bob, the chain behaves like an amplitude damping channel with some damping parameter γ. Now suppose that Alice and Bob share, say, 5 identical (uncoupled) spin chains which, for the moment, we assume all have the same damping parameter (we will see how to relax this restriction later). Consider the above protocol for the case where $k = m = 1$ (i.e., Alice and Bob are separated by a distance L, and use the 5 qubit code without concatenation). If γ is small enough, this results in the qubit being communicated with a higher fidelity than would have been achieved if only a single spin chain had been used. By using 5^k identical spin chains and concatenating the procedure k times, it is then possible to boost the fidelity to become closer and closer to unity. However, obtaining an analytic expression for the fidelity as a function of the degree of concatenation is complicated by the fact that the 5 qubit code does not preserve the structure of amplitude damping channels. In other words, the above procedure is not equivalent to sending a qubit down an effective amplitude damping channel with reduced damping parameter. Fortunately, conjugating any quantum channel by a random single-qubit Clifford operation (i.e., operations from the normalizer of the Pauli group)—a process known as channel twirling [20]—turns that channel into a depolarizing channel, whose structure is preserved by the 5 qubit code. For an amplitude damping channel with parameter γ, the twirling results in a depolarizing channel with parameter $p = [\gamma + 2(1 - \sqrt{1-\gamma})]/3$. If each of the depolarizing channels has the same parameter p, it is simple to calculate that the above procedure is equivalent to sending a single qubit down an effective depolarizing channel with parameter p_1, where p_1 is strictly less than $\frac{15}{2}p^2$. Concatenating the procedure k times produces a depolarizing channel with parameter $p_k < \frac{2}{15} \times (\frac{15}{2}p)^{2^k}$.

We can view the entire concatenated procedure as sending a qubit down a block of 5^k spin chains, each of length L. If Alice and Bob are now separated by a distance mL, they can compose m of these blocks together, one after another, to form a longer block of total length mL. Each constituent sub-block is a depolarizing channel with parameter p_k, and it follows that the composition of m such channels is itself a depolarizing channel with parameter $p_{\text{total}} = 1 - (1 - p_k)^m$, which is bounded by

$$p_{\text{total}} < 1 - \left[1 - \frac{2}{15}\left(\frac{15}{2}p\right)^{2^k}\right]^m. \tag{5}$$

Provided that $p < 2/15$ and the number of chains scales polylogarithmically with m as

$$n = 5^k \sim \left(\ln\frac{m}{\epsilon}\right)^3 \left(-\ln\frac{15p}{2}\right)^{-3}, \tag{6}$$

where $\epsilon > 0$, and the tilde ($\sim$) indicates that $n = 5^k$ can be chosen to be the smallest power of 5 greater than or equal to the right hand side of (6), we have that

$$5^k > \left(\frac{\ln[\frac{2}{15}(1+\frac{m}{\epsilon})]}{-\ln\frac{15p}{2}}\right)^{\log_2 5} \tag{7}$$

$$\Rightarrow 2^k > \left(\frac{\ln[\frac{15}{2}(1-e^{-\epsilon/m})]}{\ln\frac{15p}{2}}\right) \tag{8}$$

$$\Rightarrow p_{\text{total}} < 1 - e^{-\epsilon} < \epsilon. \tag{9}$$

Thus, provided that the parameter p of the fundamental depolarizing channels is less than $2/15$, we are free to choose any m, arbitrarily large, and any ϵ, however close to zero, and the overall channel will have parameter $p_{\text{total}} < \epsilon$. The average fidelity of sending a qubit down this channel is then $f > 1 - \epsilon/2$. Since we require that $p < 2/15$, given a degree of disorder δ, we choose a value of L such that this is true, at least with high probability. In other words, the degree of disorder fixes the maximum possible length of our sub-blocks, or equivalently, the length before which we must error correct.

Let us make a number of observations. In this proof, we assume that each channel has an identical error parameter. This is, of course, an unreasonable assumption since the errors are stochastic. However, the protocol still leads to polylogarithmic scaling if, for example, the channels all have error parameter p below some given threshold. An interesting question for the future is to analyze how well the protocol succeeds if one only knows the distribution of channel parameters.

Although we have focused here on diagonal and off-diagonal disorder, we reiterate that the same holds true more generally, provided the disorder does not couple subspaces of different numbers of excitations. It will be realized that our protocol deals with rather more general errors arising in quantum communication in spin chains (e.g. decoherence) and not just those due to disorder.

From an experimental point of view, proposals have been put forward for chains of superconducting qubits [21,22] and trapped atoms [23] to be used as the spin chains for quantum state transfer. Although the implementation of our proposed scheme is technologically challenging, manipulations of this type are likely required in any future quantum computing technology. Also, the individual elements that comprise our scheme have largely already been implemented in a variety of experimental domains. In particular, the 5 qubit code was implemented in [24], a form of concatenated error correction was demonstrated experimentally in [25], and channel twirling was recently carried out in [26].

Finally, we observe that our results might be interesting in the context of solid state physics. We have shown that parallel disordered one-dimensional spin chains can support high fidelity ''conduction'' of quantum information over arbitrary distances, the number of required chains scaling only polylogarithmically with the distance. This is perhaps at odds with the intuition one might have from the fact that in one dimension, disorder inevitably prevents propagation. Of course, there is no true contradiction here. Our system is not strictly one-dimensional, but is quasi-one-dimensional; the error correction leads to subtle coupling of the separate chains. However, we believe that our techniques from quantum information may offer new insights into localization in solid state systems.

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