Causality in Condensates: Gray Solitons as Relics of BEC Formation

Wojciech H. Zurek
Theory Division, LANL, MS-B213, Los Alamos, New Mexico 87545, USA
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Symmetry breaking during phase transitions can lead to the formation of topological defects (such as vortex lines in superfluids). However, the usually studied Bose-Einstein condensates (BECs) have the shape of a cigar, a geometry that impedes vortex formation, survival, and detection. I show that, in elongated traps, one can expect the formation of *gray solitons* (long-lived, nontopological ''phase defects'') as a result of the same mechanism. Their number will rise approximately in proportion to the transition rate. This steep rise is due to the increasing size of the region of the BEC cigar where the phase of the condensate wave function is chosen locally (rather than passed on from the already formed BEC).

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Phase transitions are usually studied as equilibrium phenomena. However, as a consequence of the critical slowing-down, second-order phase transitions depart from equilibrium near the critical point, where the new broken symmetry phase is chosen. Hence, that choice must be made locally, within regions that can dynamically ''agree'' on how to break symmetry. Cosmology offers a well-known example: As pointed out by Kibble [1], relativistic causality alone limits the size of domains over which symmetry breaking can be coordinated. As a consequence, topological defects such as monopoles, cosmic strings, and domain walls can form.

In laboratory phase transitions, relativistic causality does not provide useful estimates of the domain size with the approximately uniform new phase and, hence, does not lead to predictions of defect density. One can, however, estimate the domain size by appealing to universality of second-order phase transitions [2]: Symmetry breaking is coordinated by the dynamics of the order parameter. In the vicinity of second-order transitions, critical slowing-down implies that the relaxation time (which determines the reflexes of the system) and the healing length (which sets the scale on which its order parameter ''heals,'' i.e., returns to its equilibrium value) diverge as

$$\tau = \tau_0/|\epsilon|^{\nu z}, \tag{1}$$

$$\xi = \xi_0/|\epsilon|^{\nu}. \tag{2}$$

Above, τ_0 and ξ_0 depend on the microphysics, while the critical exponents ν and z define the universality class of the transition, and ϵ is the relative temperature

$$\epsilon = \frac{T_C - T}{T_C}, \tag{3}$$

with T_C the critical temperature.

Taking the ratio of ξ and τ, one obtains:

$$v = (\xi_0/\tau_0)|\epsilon|^{-(\nu-\nu z)} = v_0|\epsilon|^{\nu(z-1)}. \tag{4}$$

This is the speed of perturbations of the order parameter. The resulting sonic horizon plays a key role.

Divergence of the healing length was recently observed in measurements of phase coherence *above* the Bose-Einstein condensate (BEC) critical point [3], demonstrating that the phase of the condensate wave function is becoming coherent over distances that increase as the critical point is approached from above, as expected from Eq. (2). If the critical region was traversed infinitesimally slowly, all of the newly created BECs would have a single coherent phase. However, when the transition is accomplished at a finite rate, critical slowing-down, Eq. (1), intervenes: As its reflexes deteriorate, the phase of the order parameter cannot establish coherence over scales larger than the sonic horizon.

In the usual discussions of topological defect formation [2,4], one first calculates the instant $\hat{t}$ at which the system ceases to follow the externally imposed variation of its parameters by comparing the time scale $\epsilon/\dot{\epsilon}$ at which relative temperature changes to the relaxation time:

$$\tau(\hat{t}) = \epsilon(\hat{t})/\dot{\epsilon}(\hat{t}). \tag{5}$$

To obtain $\hat{t}$, we need the dependence of ϵ on t. We assume that it is linear, parametrized by quench time τ_Q:

$$\epsilon = t/\tau_Q. \tag{6}$$

The system adjusts its state adiabatically as long as the imposed rate of change is slow compared to its reflexes given by the inverse of τ, Eq. (1). The transition from the adiabatic to impulse behavior happens at $\hat{t}$ given by Eq. (5), i.e., $\tau_0|\frac{\hat{t}}{\tau_Q}|^{-\nu z} = \hat{t}$. So, the order parameter ''freezes'' when the relaxation time and $\hat{t}$ coincide;

$$\hat{t} = (\tau_0\tau_Q^{\nu z})^{1/(1+\nu z)} = \hat{\tau}. \tag{7}$$

The order parameter will resume evolution only $\hat{t}$ after the critical point is passed. The scale of the fluctuations (reported in Ref. [3]) that seed structures (such as topological defects) in the broken symmetry BEC phase [2,4–6] is thus established at $\hat{t}$, i.e., at the relative temperature:

$$\hat{\epsilon} = \left(\frac{\tau_0}{\tau_Q}\right)^{1/(1+\nu z)}. \tag{8}$$

The scale given by the corresponding healing length

$$\hat{\xi} = \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{\nu/(1+\nu z)} \tag{9}$$

determines the density of defects. The phase of the newly formed BEC wave function will be coherent on scales $\sim\hat{\xi}$. Therefore, one expects a defect fragment (e.g., one section of a vortex line) per $\hat{\xi}$-sized domain [1]. In a homogeneous 3D quench, this leads to a vortex line density of $\sim\hat{\xi}^{-2}$ [2,4,7], in accord with most of the experimental evidence [6], including BECs [8,9]. It is confirmed and refined by numerics [10], which also indicate that there is typically less than one defect fragment per $\hat{\xi}$-sized domain: Rather, a defect fragment is usually found in a larger $f\hat{\xi}$-sized region, where τ_Q-independent f is set by the microphysics of the transition. The factor f can be greater than 1, and $f \sim 10$ are common [10]. The density of defects created by phase transitions is the best known (but not the only) prediction of this ''Kibble-Zurek mechanism'' (KZM).

In the inhomogeneous case (e.g., effectively 1D trap), the situation is different: The gas density (and, hence, local critical temperature T_C) depends on location. Thus, even when T drops uniformly due to evaporative cooling, the gas will reach local critical temperature $T_C(\vec{r})$ at different instants: $\epsilon(\vec{r}_F, t_F) = 0$ defines the *front of the transition* $\vec{r}_F(t_F)$ as it spreads through the cigar. So the critical front will appear at t_F that depends on location $\vec{r}_F$.

Before the evaporative cooling, the local density is [11]

$$\rho(\vec{r}) = \rho_0 \exp[-\beta V(\vec{r})]. \tag{10}$$

Above $V(\vec{r})$ is the (typically, harmonic) trap potential and $\beta = 1/k_B T$. Einstein's condition for BEC formation involves density and de Broglie wavelength, $\rho\lambda_{\mathrm{dB}}^3(T_C) \approx 2.61$. In elongated traps, one can in effect eliminate transverse dimensions [12]. This implies a *local* $T_C(x)$:

$$T_C(x) \simeq \frac{2\pi\hbar^2}{mk_B}\left(\frac{\rho(x)}{2.61}\right)^{2/3}, \tag{11}$$

where m is the mass of bosons, while $\hbar$ and k_B are Planck and Boltzmann constants, respectively.

When anywhere in a large effectively 1D harmonic trap the temperature falls below local $T_C(x)$ in a region large compared to the healing length, the condensate will begin to form. We assume that cooling decreases T uniformly, so that

$$T(t) = T_C(0)\left(1 - \frac{t}{\tau_Q}\right) \tag{12}$$

everywhere in the trap. Therefore, front coordinates x_F and t_F are related by the equation $\epsilon(x_F, t_F) = 0$, or

$$\frac{t_F}{\tau_Q} = 1 - \frac{T_C(x_F)}{T_C(0)}. \tag{13}$$

So the condensate can form first in a healing length size domain near $x = 0$, where the potential is deepest. In an infinitesimally slow quench, that initial seed would grow to occupy the whole trap. But this cannot happen when the quench is so fast that regions far away from the center quickly attain temperatures far below the local $T_C(x)$: They will begin to form BECs independently, from local seeds and with locally selected phases.

The phase of the newly formed BEC wave function can be then either communicated along x or selected at different points of the trap independently (as would be the case in a homogeneous quench). What actually happens is decided by causality and depends on the comparison of the causal horizon defined by the relevant sound velocity [Eq. (4)] and the velocity of the front:

$$v_F = \left|\frac{dx_F}{dt_F}\right| = \frac{T_C(0)}{\tau_Q}\left|\frac{dT_C(x)}{dx_F}\right|^{-1}. \tag{14}$$

The speed of the front is infinite at the center of the trap where $V(x)$ has its minimum and drops with the inverse of the gradient of the critical temperature. The perturbations travel distance $\sim\hat{\xi}$ over time $\hat{t}$. So, the relevant speed of sound corresponds to the freeze-out $\hat{\epsilon}$:

$$\hat{v} = \frac{\hat{\xi}}{\hat{\tau}} = \frac{\xi_0}{\tau_0}\left(\frac{\tau_0}{\tau_Q}\right)^{\nu(z-1)/(1+\nu z)}. \tag{15}$$

The role of $\hat{v}$ and its sonic horizon emerged in discussions of vortex formation in ^{3}He superfluid. These experiments start with a cigar-shaped bubble heated above the critical point [13] which quickly cools to the temperature of the surrounding ^{3}He superfluid. One might have expected that superfluid on the outside of the bubble will impose the (uniform) phase of its wave function on the cooling ''cigar.'' That this need not happen was noted in Ref. [14]: When the front $T(x) = T_C(x)$ spreads faster than $\hat{v}$, the phase of the newly formed condensate is chosen locally. Subsequent studies [15] confirmed that when the front velocity v_F exceeds $\hat{v}$, symmetry breaking happens as in a homogeneous transition, and defects appear with density inferred from $\hat{\xi}$. However, when $\hat{v} > v_F$, a preexisting condensate propagates its phase into the newly forming regions, and topological defects do not form.

In the quasi-1D traps, one does not expect to see vortices as, at formation, their $\hat{\xi}$-sized cores barely fit inside the cigar, so, even if they form, they can easily escape. Vortices do form in quasi-2D pancake traps [8,9]. But in the effectively 1D geometry there is a stable defect related to phase nonuniformity—the gray soliton [16]. It corresponds to a solution of the Gross-Pitaevski equation and describes a localized (healing length scale) nonuniformity of the BEC phase and a corresponding depletion of the condensate density. The solitons are not topological: The phase change across the soliton can be arbitrary but, far away (more than a healing length ξ away from the soliton), it asymptotes to a constant value. Its change by π yields a *dark* soliton, which causes complete depletion of the BEC density at its center. Dark solitons are stationary, but *gray* solitons, with phase change less than π and a smaller depletion of central density (hence "gray" in their name), move along the BEC cigar with velocities set by the local density depletion. When they arrive at the point where BEC density is lower, they become (locally) "dark," stop, and are reflected. Gray solitons were seen oscillating in this manner along BEC cigars [17].

We can expect that nonuniformities of phase left by the BEC formation will give rise to gray solitons. Using the KZM, we can estimate the density of phase jumps caused by the BEC formation. Thus, we can also estimate the density of gray solitons in a homogeneous region and their total number left in the trap by the phase transition into BEC. To this end, we compute local "freeze-out" values of $\hat{\xi}$, $\hat{\tau}$, and $\hat{v}$ using the local rate of change of ϵ:

$$\frac{d\epsilon(x,t)}{dt}\bigg|_x = \frac{T_C(0)}{T_C(x)}\frac{1}{\tau_Q} = \frac{1}{\tau_Q(x)}. \tag{16}$$

This defines effective local quench time

$$\tau_Q(x) = \tau_Q\frac{T_C(x)}{T_C(0)}, \tag{17}$$

which in turn yields local relative temperature

$$\epsilon(x,t) = \frac{t - t_F(x)}{\tau_Q(x)}. \tag{18}$$

Now one can proceed as usual and compute local $\hat{t}$:

$$\hat{t}_x = \left[\tau_0\left(\tau_Q\frac{T_C(x)}{T_C(0)}\right)^{\nu z}\right]^{1/(1+\nu z)} = [\tau_0\tau_Q^{\nu z}(x)]^{1/(1+\nu z)}. \tag{19}$$

Note that this $\hat{t}_x$ gives the time interval to the instant $t_F(x)$ at which the critical point is reached at the location x, and Eq. (16) is satisfied. This corresponds to the local

$$\hat{\epsilon}_x = \frac{\hat{t}_x}{\tau_Q(x)} = \left(\frac{\tau_0}{\tau_Q(x)}\right)^{1/(1+\nu z)}. \tag{20}$$

We have now all of the ingredients to calculate the local frozen out healing length:

$$\hat{\xi}_x = \frac{\xi_0}{\hat{\epsilon}_x^{\nu}} = \xi_0\left(\frac{\tau_Q(x)}{\tau_0}\right)^{\nu/(1+\nu z)}. \tag{21}$$

Local velocity at the freeze-out is then a function of x:

$$\hat{v}_x = \frac{\hat{\xi}_x}{\hat{\tau}(x)} = \frac{\xi_0}{\tau_0}\left(\frac{\tau_0}{\tau_Q(x)}\right)^{\nu(z-1)/(1+\nu z)}. \tag{22}$$

These estimates are essentially the same as for the homogeneous case: Key modification enters through the locally defined $\tau_Q(x)$ [Eq. (17)].

These predictions apply to the central part of the cigar where the critical front spreads faster than $\hat{v}$—than the velocity of the perturbations of the order parameter. The region where the above quasihomogeneous quench predictions are accurate must therefore satisfy $v_F > \hat{v}_x$. In view of our above discussion, this leads to

$$\frac{T_C(0)}{\tau_Q}\left|\frac{dT_C(x)}{dx}\right|^{-1} > \frac{\xi_0}{\tau_0}\left(\frac{\tau_0}{\tau_Q(x)}\right)^{\nu(z-1)/(1+\nu z)}. \tag{23}$$

When $V(x) = \frac{m\omega^2x^2}{2}$, $T_C(x)$ [Eq. (11)] is a Gaussian:

$$T_C(x) = T_C(0)e^{-x^2/2\Delta^2}, \tag{24}$$

where $\Delta^{-2} = \frac{2}{3}\beta m\omega^2$, and we ignored variations perpendicular to the long axis. Inequality (23) leads to

$$|\hat{X}| < \frac{\Delta^2}{\xi_0}\left(\frac{\tau_0}{\tau_Q}\right)^{(1+\nu)/(1+\nu z)} e^{-\hat{X}^2/(1+\nu z)\Delta^2}. \tag{25}$$

This inequality determines the size of the section $[-\hat{X}, \hat{X}]$ of the cigar where $v_F > \hat{v}$, and the motion of the critical point is supersonic. There the quench is effectively homogeneous, and defects (including solitons) will appear with separations given by the local $\hat{\xi}$ (see Fig. 1).

The equation for $\hat{X}$ is simple, but it is transcendental. We focus on the case where $\hat{X} < \Delta$. Then the exponent in Eq. (25) can be approximated by unity, which leads to

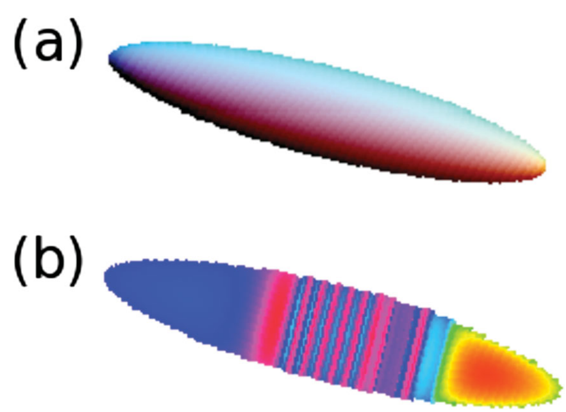

FIG. 1 (color). Formation of gray solitons in a cigar-shaped Bose-Einstein condensate. (a) Isodensity contour in the trapped gas. As evaporative cooling proceeds, critical temperature is first reached in the center of the trap. That is where the condensate will form first. When cooling is sufficiently slow, this initial seed grows and imposes its selection of the condensate wave function phase on the whole cigar, and no gray solitons are created by the quench. (b) As is seen in the schematic color plot of the wave function phase in a cross section of a BEC cigar, the situation changes when the BEC phase front—the location where the decreasing temperature is instantaneously equal to the local critical temperature (set by the local density via Einstein's condition)—moves faster than the velocity $\hat{v}$ with which perturbations of the emerging order parameter can spread. In this case, regions of size $\hat{\xi}$, the relevant healing length, select the phase of the BEC wave function independently. The front velocity v_F is infinite at $x = 0$ but falls rapidly with the distance x from the center. The condensate phase will be selected randomly by the symmetry breaking process in regions where $\hat{v} < v_F$. Such random phase distribution provides seeds for gray solitons. The phase front moves much less rapidly in the narrow direction of the cigar, so phases selected near the axis spread sideways, resulting in phase stripe pattern seen above in the schematic view of the BEC cigar.

$$|\hat{X}| \approx \frac{\Delta^2}{\xi_0}\left(\frac{\tau_0}{\tau_Q}\right)^{(1+\nu)/(1+\nu z)}. \qquad (26)$$

This estimate of $\hat{X}$ is valid for slow quenches; i.e., it breaks down when $\frac{\Delta}{\xi_0}(\frac{\tau_0}{\tau_Q})^{(1+\nu)/(1+\nu z)} > 1$ but holds when

$$\tau_Q \geq \tau_0\left(\frac{\Delta}{\xi_0}\right)^{(1+\nu z)/(1+\nu)} = \tau_0\left(\frac{\Delta}{\lambda_{\mathrm{dB}}}\right)^{(1+\nu z)/(1+\nu)}. \qquad (27)$$

We assume that this is indeed the case. This focus on slow quenches is anyway prudent: Our discussion assumes that, outside of the freeze-out interval, the order parameter is at or near the equilibrium set by the relative temperature $\epsilon(t)$. Very rapid quenches could strain this assumption.

Equation (27) yields simple scaling for the total number of solitons. Note that above we have set $\xi_0 = \lambda_{\mathrm{dB}}$, the de Broglie wavelength at the critical temperature [7]. We are now ready to estimate the total number of gray solitons. We obtain it by multiplying the size of the quasihomogeneous quench region by the expected density of phase changes. This yields

$$N \approx \frac{2\hat{X}}{f\hat{\xi}} = \frac{2\Delta^2}{f\lambda_{\mathrm{dB}}^2}\left(\frac{\tau_0}{\tau_Q}\right)^{(1+2\nu)/(1+\nu z)}. \qquad (28)$$

The surprise is that the scaling of the number of solitons with the quench time scale τ_Q is so steep. For example, for the plausible values $\nu = \frac{2}{3}$ and $z = \frac{3}{2}$, we predict $\frac{1+2\nu}{1+\nu z} = \frac{7}{6}$, while for mean field $\nu = \frac{1}{2}$ and $z = 2$, the exponent $\frac{1+2\nu}{1+\nu z} = 1$. So the number of gray solitons is expected to be approximately proportional to the quench rate.

Several aspects of the above prediction deserve comment. To begin, note that we have ignored all of the aspects of the process that cannot be deduced from the universality class. They will influence the size of f. Here we include issues such as how dark a gray soliton must be to count as a soliton and other matters relevant for experiments. For instance, it is known that solitons—while they are long-lived—do not live forever. Therefore, the number of solitons will depend on their survival rates.

The calculation above also addresses the question of when the quench can produce a uniform BEC. This will happen when $\hat{\xi} \gg \hat{X}$, for quenches so slow that they produce $N \ll 1$ solitons in a trap. There is also an opposite limit of very fast quenches. We shall not address it here as it is cumbersome [e.g., transcendental Eq. (25) cannot be approximated in a way that yields a simple result]. Moreover, in order to reach it in experiments, one would need to drop the temperature very quickly throughout the trap and far below T_C at the center of the trap. For such rapid quenches, linear approximation $\epsilon = t/\tau_Q$ is likely to break down, leading to further cumbersome but trivial complications. This last comment brings one more remark: In most experimental settings, $T(t)$ will fall below T_C at the center of the trap, but this may be above $T_C(x)$ at some sufficiently large x. Our analysis applies as long as that x lies outside the central interval of $2\hat{X}$ or, more precisely, as long as the quench can be well approximated by linear relations [e.g., Eq. (12)] inside it.

We discussed formation of gray solitons in elongated traps. The experiment aimed at detecting such nontopological remnants of a BEC phase transition should be easier than experiments [9] that study formation of vortex lines (which require more than a quasi-1D geometry). It should allow one to probe the connection between causality and symmetry breaking and test scalings predicted by the Kibble-Zurek mechanism. Last but not least, we note that similar considerations will apply to traps with 2D and 3D geometry, as also in these cases one would expect only the central region of the trap to satisfy the appropriate analogue of the causality inequality (25).

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