

Quantum Noise as an Entanglement Meter

Israel Klich¹ and Leonid Levitov^{1,2}

¹*Kavli Institute for Theoretical Physics, University of California Santa Barbara, Santa Barbara, California 93106, USA*

²*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

(Received 10 June 2008; published 9 March 2009)

Transport in a quantum point contact (QPC) can be used to generate many-body entanglement of Fermi seas in the leads. A universal relation is found between the generated entanglement entropy and the fluctuations of electric current, which is valid for any protocol of driving the QPC. This relation offers a basis for direct electric measurement of entanglement entropy. In particular, by utilizing space-time duality of 1D systems, we relate electric noise generated by opening and closing the QPC periodically in time with the seminal $\mathcal{S} = \frac{1}{3} \log L$ prediction of conformal field theory.

DOI: 10.1103/PhysRevLett.102.100502

PACS numbers: 03.67.Mn, 03.65.Ud, 05.50.+q, 05.70.Jk

Recent years have witnessed a burst of interest in the phenomena of quantum entanglement, and, in particular, in entanglement entropy [1,2], a fundamental characteristic describing quantum many-body correlations between two parts of a quantum system. This quantity proved useful in analyzing quantum critical phenomena [3–5], quantum quenches [6–9], topologically ordered states [10–12], and strongly correlated systems [13]. Entanglement entropy is also of fundamental interest for quantum information theory as a measure of the resources available for quantum computation [14].

The task of measuring entanglement entropy in a system comprised of a large number of particles is substantially more challenging than detecting entanglement of few particles, such as in the recent work on entangled photons [15]. Since realistic measurement processes cannot simultaneously access all degrees of freedom in a many-body system, the quantities like the full many-body density matrix, which depends on coordinates of all particles in the system, are very difficult to measure.

As we shall see, the situation with the entanglement entropy is different. In this Letter, we establish a relation between the entropy and quantum noise in a quantum point contact (QPC) [16]. The QPC is an electron beam splitter with tunable transmission and reflection. In our approach, it serves as a door between electron reservoirs, which can be opened and closed on demand (see Fig. 1).

The entanglement entropy is dominated mostly by *charge-neutral* processes in which the leads exchange particles with no net charge transfer. Yet, somewhat surprisingly, we find that a measurement of the fluctuations of electric current flowing through the QPC is sufficient for determining the full entanglement entropy. Here, we derive the noise-entropy relation for noninteracting fermions.

The process of connecting and then disconnecting two parts of the system is a space-time dual of the setting considered in Refs. [2–5], where the many-body ground state of a translationally invariant system is analyzed using a finite region in space. In our case, a window in time is

used, $t_0 < t < t_1$, during which particles can delocalize among the reservoirs, making them entangled.

The relation between entanglement and electric noise is central for the proposals [17,18] to use current partitioning by a QPC for producing entangled particle pairs. A relation between entanglement entropy and another measurable quantity, particle number fluctuations, was considered in Ref. [19]. Generation of entanglement was also analyzed for critical Hamiltonians [6], for generic Hamiltonians [7,9], as well as for a QPC under bias voltage [20].

Here, we obtain a general relation between entanglement production and the Full Counting Statistics (FCS) [21] which describes the statistics of transmitted charge. The central quantity in the FCS approach is the generating function $\chi(\lambda) = \sum_{n=-\infty}^{\infty} P_n e^{i\lambda n}$, where P_n is the probability to transmit n charges in total. This function encodes all FCS higher moments:

$$\log \chi(\lambda) = \sum_{m=1}^{\infty} \frac{(i\lambda)^m C_m}{m!}, \quad (1)$$

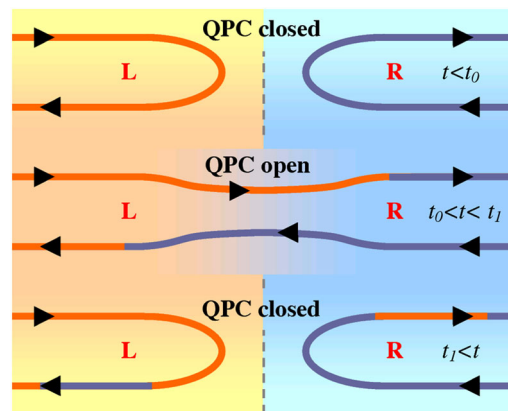


FIG. 1 (color online). Schematic of a quantum point contact (QPC) with transmission changing in time. The left and right leads are initially disconnected, then connected at $t_0 < t < t_1$, and then disconnected again. Electron transport, taking place at $t_0 < t < t_1$, leading to electron delocalization among the leads, current fluctuations, and entanglement production.

where the cumulants $C_1, C_2, C_3 \dots$ describe properties of the distribution P_n such as the mean \bar{n} , the variance $\langle(n - \bar{n})^2\rangle$, the skewness $\langle(n - \bar{n})^3\rangle$, etc. The cumulant C_2 is available from routine noise measurement. Recently, C_3 has been measured in tunnel junctions [22,23] and in QPC [24], while cumulants up to 5th order have been measured in quantum dots [25,26].

Below, we establish a universal relation between FCS (1) and the entanglement entropy generated in the QPC. Only even cumulants are shown to contribute to entropy:

$$S = \sum_{m>0} \frac{\alpha_m}{m!} C_m, \quad \alpha_m = \begin{cases} (2\pi)^m |B_m|, & m \text{ even} \\ 0, & m \text{ odd} \end{cases}, \quad (2)$$

where B_m are Bernoulli numbers [27] ($B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42} \dots$). The first few contributions are

$$S = \frac{\pi^2}{3} C_2 + \frac{\pi^4}{15} C_4 + \frac{2\pi^6}{945} C_6 + \dots \quad (3)$$

This relation, which is general and valid for arbitrary driving, can be used to determine the entanglement entropy from measured values of FCS moments. Similar relation can be derived for other quantities of interest, such as Renyi entropies and single copy entropy [28].

In particular, entanglement generated in a QPC switching on and off, as illustrated in Fig. 1, directly corresponds to the entanglement entropy found for conformal field theory, $S = \frac{c}{3} \log L$, where L is the size of window in space and c the conformal charge [2,3,5]. In this case, the current fluctuations are Gaussian ($C_{m \neq 2} = 0$), with a logarithmic variance $C_2 = \frac{1}{\pi^2} \log \frac{t-t_0}{\tau}$, where τ is a short time cutoff set by the QPC switching rapidity. Combined with Eq. (3), this gives entropy $S = \frac{1}{3} \log \frac{t-t_0}{\tau}$. Below, we discuss how this logarithmic dependence can be verified using the setup shown in Fig. 1.

Entanglement entropy is conventionally defined as the von Neumann entropy $S(\rho) = -\text{Tr} \rho \log \rho$, where ρ is the reduced density matrix of a pure quantum state, made ‘‘impure’’ by confining it to a certain space region [1,2]. In our case, the many-body state evolves as a pure state while the QPC is open (see Fig. 1), after which the reduced density matrix of the lead L is given by $\rho_L(t_1) = \text{Tr}_R[\mathbf{U}(t_1, t_0)\rho_0\mathbf{U}^\dagger(t_0, t_1)]$. Here, ρ_0 is the initial density matrix of the system, $\mathbf{U}(t_1, t_0)$ describes the many-body evolution between t_0 and t_1 , and Tr_R is a partial trace over degrees of freedom in the lead R .

Entropy production in the lead L as a result of QPC opening and closing is given by the difference

$$\Delta S = S[\rho_L(t)] - S(\rho_0), \quad (4)$$

where the last term accounts for the entropy in the initial state. Because at finite temperature both terms in Eq. (4) are proportional to the lead volume, they can be large for macroscopic leads. The increment ΔS , however, remains well defined regardless of the lead volume.

Below, we focus on the zero temperature case when ρ_0 is a pure state, described as a filled Fermi sea in the full

system $L + R$, in which case the second term in Eq. (4) vanishes, giving $\Delta S = S[\rho_L(t)]$. We associate with ρ_0 a Fermi projection operator n in the single-particle space $\langle E|n|E'\rangle = \delta_{E,E'}\theta(E_F - E)$, where E_F is the Fermi energy. The evolved system is described by a rotated Fermi projection $n_U = \mathbf{U}n\mathbf{U}^\dagger$, where \mathbf{U} is the unitary evolution of the single-particle modes.

Our first step will be to express the entropy in terms of single-particle quantities. For a generic Gaussian state, Wick’s theorem for operator products is satisfied in $L + R$, and therefore, in particular, it holds in L [29]. Therefore, the reduced density matrix ρ_L is also Gaussian: $\rho_L = Z^{-1} e^{-\tilde{H}_{ij} a_i^\dagger a_j}$ for some \tilde{H} , where i, j label the states in L . We define a single-particle quantity $m_{ij} = \text{Tr} \rho_L a_i^\dagger a_j$. For the evolved system, described by n_U , Wick’s theorem gives $m_{ij} = (n_U)_{ij}$. In what follows, it will be convenient to extend m to $L + R$ by setting

$$M = P_L n_U P_L, \quad (5)$$

with P_L a projection on the modes in L , so that $M = m$ in L and $M = 0$ in R .

Entropy can be expressed through m_{ij} for a generic Gaussian state. Because of Fermi-Dirac statistics, $m = (1 + e^{\tilde{H}})^{-1}$, which gives $\tilde{H} = \log(m^{-1} - 1)$ [29]. Extending m to M in $L + R$, we write the entropy as

$$S(\rho) = -\text{Tr}[M \log M + (1 - M) \log(1 - M)] \quad (6)$$

where now the trace is taken in the space of single-particle modes in $L + R$.

Our next step is to relate the quantity M , Eq. (5), and the FCS generating function (1) which can be expressed as a functional determinant [21]:

$$\chi(\lambda) = \det(1 - n + n\mathbf{U}^\dagger e^{i\lambda P_L} \mathbf{U} e^{-i\lambda P_L}). \quad (7)$$

This determinant must be properly regularized for infinitely deep Fermi sea [30,31]. For our purposes, we proceed to treat it as a finite matrix and obtain [32,33]

$$\chi(\lambda) = \det[(1 - M + M e^{i\lambda}) e^{-i\lambda(nP_L)_U}], \quad (8)$$

where M is the quantity (5) which defines the entropy.

Now, with the help of the relation (8) we can express the spectral density of M , Eq. (5), which lies between 0 and 1, through $\chi(\lambda)$. Indeed, changing parameter λ to $z = (1 - e^{i\lambda})^{-1}$ yields $z - M$ under the determinant: $\chi(z) = \det[(z - M) e^{-i(nP_L)_U \lambda(z)} (1 - e^{i\lambda(z)})]$. From that, we can write the spectral density of M as

$$\mu(z) = \frac{1}{\pi} \text{Im} \partial_z \log \chi(z - i0) + A\delta(z) + B\delta(z - 1), \quad (9)$$

where the coefficients A and B depend on $\dim M$, $\text{Tr}(nP_L)_U$, and C_1 . Hereafter, we ignore the delta function terms because $z = 0, 1$ do not contribute to the expression (6) for the entropy which we rewrite as

$$S = - \int_0^1 dz \mu(z) [z \log z + (1 - z) \log(1 - z)]. \quad (10)$$

Now it is straightforward to evaluate \mathcal{S} by substituting (1) into (9) and (10) with $\lambda(z) = \pi - i \log(\frac{1}{z} - 1)$, and integrating by parts over z . We obtain series $\mathcal{S} = \sum_{m=1}^{\infty} \frac{\alpha_m}{m!} C_m$, where the coefficients α_m , after changing the integration variable in (10) to $u = \frac{1}{2} \log(\frac{z}{1-z})$, take the form

$$\alpha_m = \frac{(-2)^m}{\pi} \int_{-\infty}^{\infty} du \frac{u}{\cosh^2 u} \text{Im} \left(\frac{i\pi}{2} + u \right)^m. \quad (11)$$

After shifting the contour of integration as $u \rightarrow u - i\frac{\pi}{2}$ and using an identity [34], we arrive at our main result (2).

To compute FCS for a specific driving protocol, it is convenient to return to the expression (7) and apply the Riemann-Hilbert (RH) method introduced in [30]. In this approach, one must factor the time-dependent matrix $R(t) = U^\dagger(t) e^{i\lambda P_L} U(t) e^{-i\lambda P_L}$ in (7), i.e., find matrix valued functions $X_\pm(z)$, analytic in the upper or lower half plane of complex z , respectively, such that on the real line,

$$X_+(t) = X_-(t)R(t) \quad (12)$$

with normalization $X_\pm(z) \rightarrow I$ at $|z| \rightarrow \infty$.

We consider QPC switching between the on and off states several times $t_0^{(1)} < t_1^{(1)} < \dots < t_0^{(N)} < t_1^{(N)}$. The RH problem is solvable in the case of abrupt switching because R commutes with itself at different times: $R = I$ in the off state, $R = e^{-i\lambda(P_R - P_L)}$ in the on state. The solution of the RH problem is then given by the functions

$$X_\pm(z) = \exp\left(\frac{\lambda}{2\pi} (P_R - P_L) \sum_{i=1}^N \log \frac{z - t_0^{(i)} \pm i0}{z - t_1^{(i)} \pm i0}\right). \quad (13)$$

To find the determinant (7) with these X_\pm , we use the RH method [30] to evaluate the derivative of $\log \chi$:

$$\begin{aligned} \partial_\lambda \log \chi(\lambda) &= \int \text{tr} \left(\frac{1}{2\pi i} X_+^{-1} \partial_t X_- \partial_\lambda R \right) dt = -\frac{\lambda}{2\pi^2} G, \\ G &= \sum_{i,j=1}^N \log \frac{t_1^{(i)} - t_0^{(j)}}{t_0^{(i)} - t_0^{(j)}} + \log \frac{t_1^{(i)} - t_0^{(j)}}{t_1^{(i)} - t_1^{(j)}}, \end{aligned} \quad (14)$$

where for $i = j$, the denominators must be replaced by a short-time cutoff τ . This gives Gaussian charge statistics

$$\chi(\lambda) = \exp(-\lambda^2 C_2/2), \quad C_2 = \frac{1}{2\pi^2} G. \quad (15)$$

Because in this case the only nonvanishing cumulant is C_2 , we have $\mathcal{S} = \frac{\pi^2}{3} C_2$, which gives $\mathcal{S} = \frac{1}{3} \log \frac{t_1 - t_0}{\tau}$ for a single QPC switching, in agreement with the $\mathcal{S} = \frac{1}{3} \log L$ relation [2]. The case of multiple switching provides realization of the situation studied in Ref. [5].

These predictions can be tested by measuring noise in a QPC driven by a periodic train of pulses (see Fig. 2). For N identical pulses, the relation (14) at large N yields

$$C_2(N) \approx \frac{N}{\pi^2} \log \frac{\sin \pi \nu w}{\pi \nu \tau}, \quad \nu = 1/T. \quad (16)$$

Thus, in a periodically driven QPC, there is a finite entropy production per cycle at a rate $d\mathcal{S}/dt = \frac{1}{3} \nu \log \frac{\sin \pi \nu w}{\pi \nu \tau}$. Fluctuations (16) with $C_2 \propto N$ correspond to a dc electric

noise of intensity

$$S_2 = \frac{e^2 \nu}{\pi^2} \log \frac{\sin \pi \nu w}{\pi \nu \tau} \quad (17)$$

at frequencies below ν . For a short pulsewidth w , the dependence (17) becomes $S_2 = \frac{e^2 \nu}{\pi^2} \log \frac{w}{\tau}$, identical to the entropy for a single pulse.

The result (17) must be compared with thermal noise. At a driving frequency $\nu = 500$ MHz, the effective temperature corresponding to (17) is $T_{\text{eff}} = \frac{h\nu}{\pi^2 k_B} \log \frac{\sin \pi \nu w}{\pi \nu \tau} \approx 25$ mK. In practice, it may be possible to relax the constraint due to small T_{eff} by detecting the noise (17) at frequencies somewhat higher than $k_B T_{\text{eff}}/h$, detuned from the thermal noise spectral window.

How sensitive are these results to imperfections in QPC transmission? It is straightforward to incorporate transmission $D < 1$ in the ‘‘on’’ state in the RH analysis because the matrices $R(t)$ in (7) still commute at different times. Instead of $e^{\pm i\lambda}$, the eigenvalues of the R matrix are now $e^{\pm i\lambda_*}$ with $\sin \frac{1}{2} \lambda_* = \sqrt{D} \sin \frac{1}{2} \lambda$ [35]. Making this change, we obtain

$$\chi(\lambda) = \exp(-\lambda_*^2 G/4\pi^2), \quad (18)$$

with G given by (14) as above. Because this $\chi(\lambda)$ is non-Gaussian, with nonzero higher cumulants, the simplest way to find the entropy is to use its relation with the spectral density of M , Eqs. (9) and (10). Using (18) along with the relations between λ_* , λ , and z , we find

$$\mu(z) = \frac{G}{\pi^2} \frac{D}{z(1-z)} \text{Re} \frac{|1-2z|}{\sqrt{D^2 - 4Dz(1-z)}}. \quad (19)$$

As illustrated in Fig. 3, at $D < 1$, the function $\mu(z)$ vanishes in the interval $z_- < z < z_+$, $z_\pm = \frac{1}{2}(1 \pm \sqrt{1-D})$.

The entropy, found from (10) and (19), has the same logarithmic dependence (14) on the times $t_{0,1}^{(i)}$ as above, albeit with a D -dependent prefactor. Thus, the predicted

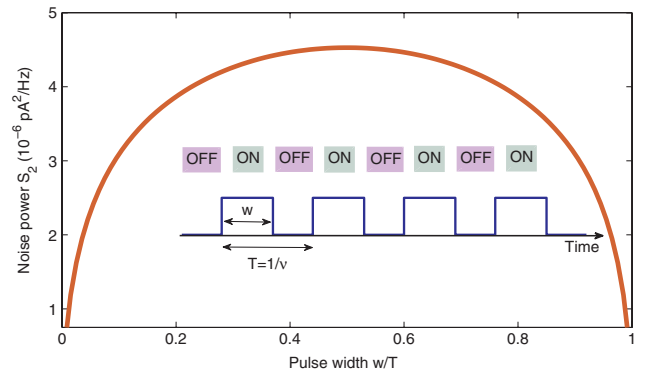


FIG. 2 (color online). Noise power (17) in a QPC driven by a pulse train vs the pulse width. Parameters used: driving frequency $\nu = 500$ MHz, short-time cutoff $\tau = 20$ ps. The noise as well as the entropy production are symmetric under $w \rightarrow T - w$. Note that at a narrow pulsewidth $w \ll T$, the dependence (17) reproduces the $\frac{1}{3} \log L$ [2] behavior of the entropy.

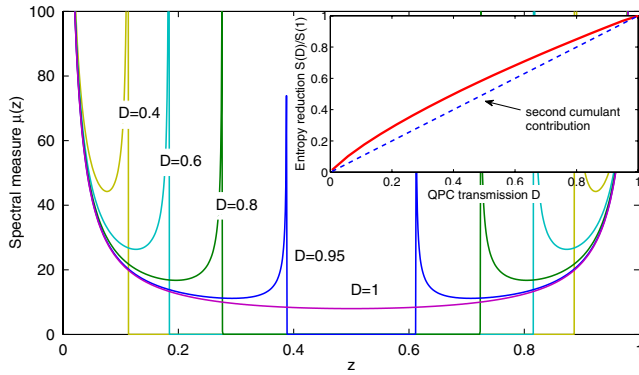


FIG. 3 (color online). The effect of imperfect transmission in the QPC. Spectral density of M , Eq. (19), scaled by $G/2\pi^2$, for a QPC driven by pulses with transmission D in the “on” state (see Fig. 2). At $D < 1$, the entropy (10) is reduced by a constant factor (inset) with the $S \sim \log \sin \pi \nu w$ dependence unchanged.

dependence $S \sim \log \sin \pi \nu w$ remains robust. The behavior of the rescaling factor $F = S(D)/S(1)$ (Fig. 3 inset) indicates that entropy reduction due to imperfect transmission in QPC can be attributed mostly to the change in the second cumulant, $C_2 = \frac{D}{2\pi^2}G$, with a relatively small correction due to higher cumulants.

In the presence of a dc voltage bias, we have $\chi(\lambda) = (1 - D + De^{i\lambda})e^{Vt/h}$ [21], whereas the entropy production rate was found to be $dS/dt = -\frac{eV}{h}[D \log D + (1 - D) \times \log(1 - D)]$ [20]. These quantities satisfy the universal relation (2), as can be seen most easily from Eq. (6).

From the quantum information perspective, it is interesting to quantify the part of the entropy accessible to local operations (i.e., respecting particle conservation in each lead) [20]. In a many-particle system, the change in entropy due to such restriction turns out to be negligibly small [36].

In summary, we derived a general relation between entanglement and noise in terms of the full counting statistics. This relation provides a new framework to investigate many-body entanglement, and, in particular, its generation in nonequilibrium quantum systems.

This work was supported in part by the National Science Foundation under Grant No. PHY05-51164 and by W.M. Keck Foundation Center for Extreme Quantum Information Theory (L.L.). We thank Carlo Beenakker and Gil Refael for useful discussions.

- [1] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D **34**, 373 (1986).
- [2] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B **424**, 443 (1994).
- [3] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).
- [4] G. Refael and J. E. Moore, Phys. Rev. Lett. **93**, 260602 (2004).

- [5] P. Calabrese and J. Cardy, J. Stat. Mech. (2004) P06002.
- [6] P. Calabrese and J. Cardy, J. Stat. Mech. (2005) P04010.
- [7] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).
- [8] M. Cramer, C. M. Dawson, J. Eisert, and T. J. Osborne, Phys. Rev. Lett. **100**, 030602 (2008).
- [9] J. Eisert and T. J. Osborne, Phys. Rev. Lett. **97**, 150404 (2006).
- [10] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
- [11] M. Levin and X. G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
- [12] S. Dong, E. Fradkin, R. G. Leigh, and S. Nowling, J. High Energy Phys. **05** (2008) 016.
- [13] F. Verstraete, D. Porras, and J. I. Cirac, Phys. Rev. Lett. **93**, 227205 (2004).
- [14] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A **53**, 2046 (1996).
- [15] S. P. Walborn *et al.*, Nature (London) **440**, 1022 (2006).
- [16] B. J. van Wees *et al.*, Phys. Rev. Lett. **60**, 848 (1988).
- [17] C. W. J. Beenakker, C. Emery, M. Kindermann, and J. L. van Velsen, Phys. Rev. Lett. **91**, 147901 (2003).
- [18] P. Samuelsson, E. V. Sukhorukov, and M. Buttiker, Phys. Rev. Lett. **91**, 157002 (2003).
- [19] I. Klich, G. Refael, and A. Silva, Phys. Rev. A **74**, 032306 (2006).
- [20] C. W. J. Beenakker, Proc. Int. School Phys. E. Fermi (IOS Press, Amsterdam, 2006), Vol. 162.
- [21] L. S. Levitov and G. B. Lesovik, JETP Lett. **58**, 230 (1993).
- [22] B. Reulet, J. Senzier, and D. E. Prober, Phys. Rev. Lett. **91**, 196601 (2003).
- [23] Yu. Bomze *et al.*, Phys. Rev. Lett. **95**, 176601 (2005).
- [24] G. Gershon, Yu. Bomze, E. V. Sukhorukov, and M. Reznikov, Phys. Rev. Lett. **101**, 016803 (2008).
- [25] T. Fujisawa, T. Hayashi, R. Tomita, and Y. Hirayama, Science **312**, 1634 (2006).
- [26] S. Gustavsson *et al.*, Phys. Rev. Lett. **96**, 076605 (2006).
- [27] The Bernoulli numbers B_m are defined by the generating function $\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$. Asymptotically, $|B_n| \approx \frac{2n!}{(2\pi)^n}$ for large even n , which shows that the coefficients in (2) stay bounded for high order cumulants.
- [28] J. Eisert and M. Cramer, Phys. Rev. A **72**, 042112 (2005); This paper uses a spectral representation similar to our Eq. (10).
- [29] I. Peschel, J. Phys. A **36**, L205 (2003).
- [30] B. A. Muzykantskii and Y. Adamov, Phys. Rev. B **68**, 155304 (2003).
- [31] J. E. Avron, S. Bachmann, G. M. Graf and I. Klich, Commun. Math. Phys. **280**, 807 (2008).
- [32] For derivation of Eq. (8), see I. Klich, L. S. Levitov, arXiv:0804.1377v1, Eqs. (11), (12), and (13).
- [33] Our derivation of (8) is similar to that used in: A. G. Abanov and D. A. Ivanov, Phys. Rev. Lett. **100**, 086602 (2008), where constraints on $\chi(\lambda)$ due to the spectrum of M being inside the interval $0 < z < 1$ are also discussed.
- [34] We use the integral $\int_0^{\infty} \frac{u^{2m} du}{\sinh^2 u} = \pi^{2m} |B_{2m}|$, Eq. (3.525) in: I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1980).
- [35] L. S. Levitov and G. B. Lesovik, arXiv:cond-mat/9401004.
- [36] I. Klich and L. S. Levitov, arXiv:0812.0006.