Towards Loop Quantum Gravity without the Time Gauge

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The Hamiltonian formulation of the Holst action is reviewed and it provides a solution of second-class constraints corresponding to a generic local Lorentz frame. Within this scheme the form of rotation constraints can be reduced to a Gauss-like one by a proper generalization of Ashtekar-Barbero-Immirzi connections. This result emphasizes that the loop quantum gravity quantization procedure can be applied when the time-gauge condition does not stand.

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Introduction.—The only consistent procedure to quantize nonperturbatively the gravitational field is given by loop quantum gravity (LQG) (see [1] for some reviews). One of the most impressive results within this framework consists in the discreteness of geometrical operator spectra on a kinematical level [2], which has been recently confirmed by a suitable reduction of a spin-foam model [3]. In view of this discreteness it must be demonstrated that the Lorentz invariance is still preserved on a quantum level. As far as Lorentz transformations are concerned, one must distinguish Lorentz coordinate transformations, which act on the space-time metric, from transformations on the tangent space. While the invariance under the former is insured by general covariance [4] (see also [5]), the role of the latter is controversial in LQG. This is because the local Lorentz frame is fixed by the time-gauge condition before quantizing.

The Hamiltonian formulation of the Holst action [6] in a generic Lorentz frame was given by Barros e Sa in [7]. Such an analysis outlines the appearance of some second-class constraints and provides a solution for them, so that only first-class constraints remain. Since the Barbero-Immirzi formulation is found by fixing the time gauge, this formulation must be related by a gauge transformation to a boosted one. However, it remains to demonstrate that the boost invariance is preserved on quantum level too. Indeed, there are examples where first-class constraints are not associated with quantum symmetries [8]. In this respect a formulation free of the time gauge could be highly nontrivial.

Furthermore, even though the boost invariance holds, the development of a quantum theory without the time gauge can give insight on how Lorentz transformations act on a discrete space structure, especially in view of introducing elementary particles as fermion matter.

A totally covariant approach to the quantization of gravity was proposed by Alexandrov in [9], where Gauss constraints for the Lorentz group were inferred. Nevertheless, since in general connections do not commute, the quantization of that model is still an open issue.

In this work, we provide a solution of second-class constraints and we develop the corresponding Hamiltonian formulation. At first we investigate the implications of fixing a generic Lorentz frame by assigning time-independent velocity components χ_a with respect to spatial hypersurfaces. In [7] only the case $\chi_a = 0$ is considered in detail, while choosing different gauge conditions implies solving the boost constraints with respect to momenta conjugate to χ_a. In particular, a redefinition of variables is given such that the rotation constraints have a closed algebra and they take the form of Gauss constraints.

When χ_a depends on time, their dynamical role must be considered and the gauge fixing is no more allowed. Nevertheless, as far as we deal with a canonical formalism for χ_a (which implements boost invariance) we recognize that their conjugate momenta are constrained to vanish. Therefore the full dynamical information is contained into the dependence on $SU(2)$ gauge connections.

Hence, the quantization can be performed also without any gauge fixing and no modification arises with respect to the case when the time gauge holds. For instance, the action of the area operator is investigated and the independence on χ_a of the corresponding spectrum is outlined. Within this approach no expansion in the boost parameters is performed; thus, our results extend and confirm on a different level those ones in [4].

The organization of the Letter is as follows: the Hamiltonian formulation of the Holst action is reviewed and second-class constraints are solved. Hence, in the analysis of rotation, constraints are performed and the

gauge structure is inferred in a time-independent frame. The extension to time-dependent frames is addressed, such that the quantization without gauge fixing and the properties of the area spectrum are described. The comparison with covariant theory is outlined. Finally, brief concluding remarks follow.

Hamiltonian formulation.—The action of general relativity with the Holst modification [6] takes the following form (in units $8\pi G = 1$)

$$S = \int \sqrt{-g} e^{\mu}_A e^{\nu}_B R^{CD}_{\mu\nu}(\omega^{FG}_{\mu})^{\gamma} p^{AB}{}_{CD}, \qquad (1)$$

g being the determinant of the metric tensor $g_{\mu\nu}$ with four-bein vectors e^A_{μ} and spinor connections ω^{AB}_{μ}, while the expressions for $R^{AB}_{\mu\nu}$ and $^{\gamma}p^{AB}{}_{CD}$ are

$$R^{AB}_{\mu\nu} = \partial_{[\mu}\omega^{AB}_{\nu]} + \omega^A{}_{C[\mu}\omega^{CB}_{\nu]},$$

$$^{\gamma}p^{AB}{}_{CD} = \delta^{AB}{}_{CD} - \frac{1}{2\gamma}\epsilon^{AB}{}_{CD}. \qquad (2)$$

Here γ is the Immirzi parameter. By a Legendre transformation, conjugate momenta $^{\gamma}\pi^{\mu}_{AB}$ can be defined. One easily recognizes that $^{\gamma}\pi^t_{AB} = 0$, so ω^{AB}_t does not enter into the dynamical description. Let us now introduce π^i_{AB}, such that $^{\gamma}\pi^i_{AB} = ^{\gamma}p^{CD}{}_{AB}\pi^i_{CD}$, whose geometrical interpretation is much more clear than $^{\gamma}\pi^i_{AB}$, since $\pi^i_{AB} = 2\sqrt{-g}e^t_{[A}e^i_{B]}$.

The Hamiltonian density turns out to be a linear combination of the following constraints

$$H = \pi^i_{CF}\pi^{jF}{}_D{}^{\gamma}p^{CD}{}_{AB}R^{AB}_{ij} = 0$$

$$H_i = {}^{\gamma}p_{AB}{}^{CD}\pi^j_{CD}R^{AB}_{ij} = 0$$

$$G_{AB} = D_i\pi^i_{AB} = \partial_i\pi^i_{AB} - 2\omega_{[A}{}^C{}_i\pi^i_{|C|B]} = 0 \qquad (3)$$

$$C^{ij} = \epsilon^{ABCD}\pi^{(i}_{AB}\pi^{j)}_{CD} = 0$$

$$D^{ij} = \epsilon^{ABCD}\pi^k_{AF}\pi^{(iF}{}_B D_k\pi^{j)}_{CD} = 0.$$

In this scenario, H and H_i denote the super-Hamiltonian and the supermomentum, respectively, while G_{AB} are Gauss constraints of the Lorentz symmetry. Other constraints are C^{ij} and D^{ij}, with the latter coming out as secondary ones from the former.

The full set of constraints is second class, because $\{C^{ij}, D^{kl}\}$ and $\{D^{ij}, D^{kl}\}$ do not vanish on the constraint hypersurfaces. Hence, C^{ij} and D^{kl} are not associated with any gauge symmetry.

Solution of second-class constraints.—A generic set of 4-bein vectors can be written as

$$e^0 = Ndt + \chi_a E^a_i dx^i, \qquad e^a = E^a_i N^i dt + E^a_i dx^i. \quad (4)$$

It is worth noting the role of χ_a variables, which give the velocity components of the frame $\{e^a\}$ with respect to spatial hypersurfaces [10]. The time-gauge condition consists in $\chi_a = 0$ and once adopted boost degrees of freedom are frozen.

In what follows, we will denote π^i_{0b} with π^i_b. Let us now introduce the inverses of π^i_a ($\pi^a_i\pi^i_b = \delta^a_b$), by which the three metric can be written as $h_{ij} = \frac{1}{\pi}T^{-1}_{ab}\pi^a_i\pi^b_j$, with π the determinant of π^a_i and $T^{-1}_{ab} = \eta_{ab} + \chi_a\chi_b$. Defining $\pi\omega_a{}^b{}_i = \frac{1}{\pi^{1/2}}\pi^{b3}_l\nabla_i(\pi^{1/2}\pi^l_a)$, where $^3\nabla_i$ is the covariant derivative associated with h_{ij}, a solution to C^{ij} and D^{ij} is given by

$$\pi^i_{ab} = 2\chi_{[a}\pi^i_{b]},$$

$$\omega_a{}^b{}_i = \pi\omega_a{}^c{}_i T^{-1b}_c + \chi_a\omega^{0b}{}_i + \chi^b(\omega_a{}^0{}_i - \partial_i\chi_a). \qquad (5)$$

We want to emphasize that conditions above are solutions of second-class constraints, without any restriction on χ_a; hence, at this level the Lorentz frame is not fixed at all.

Rotation constraints in a fixed local Lorentz frame.—We will denote Ω the subspace defined by conditions (5) and we take $\{\omega^{0a}{}_i, \pi^i_a\}$ as coordinates on it. At this level we require $\partial_t\chi_a = 0$, such that no evolutionary character is given to χ_a and they can be treated as parameters. Since second-class constraints have been solved, the symplectic structure on Ω is nontrivial. For instance, $\omega^{0a}{}_i$ do not commute among each other [11]. Within this scheme, the Lorentz frame has been fixed. In fact, $D_i\pi^i_a = \chi^b D_i\pi^i_{ab}$, which means that the constraints associated to boosts become redundant.

Let us now introduce the "densitized" 3-bein of the spatial metric h_{ij}, which are conjugate momenta in the standard LQG formulation within the time gauge [1] and whose expression here reads as follows

$$\tilde{\pi}^i_a = S^b_a\pi^i_b,$$

$$S^a_b = \sqrt{1+\chi^2}\delta^a_b + \frac{1 - \sqrt{1+\chi^2}}{\chi^2}\chi_a\chi_b. \qquad (6)$$

Hence, S^a_b is a map from momenta π^i_a to 3-bein vectors of the metric on spatial hypersurfaces and it actually represents the action of the boost transformations.

Starting from the rotation constraints we sum up a vanishing contribution and multiply the result times S^a_b, so finding

$$G_a = \partial_i\tilde{\pi}^i_a + \gamma\epsilon_{ab}{}^c\tilde{A}^b_i\tilde{\pi}^i_c = 0; \qquad (7)$$

i.e., $SU(2)$ Gauss constraints are inferred also without the time-gauge condition.

Connections $\tilde{A}^a_i$ are given by

$$\tilde{A}^a_i = S^{-1a}_b\left(A^b_i + \frac{2 + \chi^2 - 2\sqrt{1+\chi^2}}{2\gamma\chi^2}\epsilon^{abc}\partial_i\chi_b\chi_c\right), \quad (8)$$

where the A^a_i variables take the following expression

$$A^a_i = (1 + \chi^2)T^{ac}(\omega_{0ci} + {}^{\pi}D_i\chi_c) - \frac{1}{2\gamma}\epsilon^a_{cd}{}^{\pi}\omega^{cf}{}_i T^{-1d}_f. \qquad (9)$$

As far as the symplectic structure in terms of new variables is concerned, it comes out to be the canonical

one, i.e., $\{\tilde{A}_i^a(t, x), \tilde{\pi}_b^j(t, y)\} = \delta_b^a \delta_i^j \delta^3(x - y)$, while the others vanish. Furthermore, it can be demonstrated that on the hypersurfaces (5) a canonical transformation maps $\{\omega_i^{AB}, \pi_{CD}^j\}$ into $\{\tilde{A}_i^a, \tilde{\pi}_b^j\}$ when $\partial_t \chi_a = 0$. These features enforce the interpretation of $\tilde{A}_i^a$ as the extension of Barbero-Immirzi connections to a generic time-independent Lorentz frame.

The same result can be obtained within the Barros e Sa framework [7]. That formulation was based on solving second-class constraints without fixing the boost symmetry. This implies that additional dynamical variables are present, i.e., χ_a and their conjugate momenta π^a, and that the vanishing of the boost constraints G_a^{boost} provides three independent conditions. However, since π^a appear linearly into such constraints, $G_a^{\text{boost}} = 0$ can be solved with respect to them. If the obtained expressions are inserted into the rotation constraints, then we get conditions equivalent to Eqs. (7). In fact, variables A_i^a differ from the ones of Barros e Sa, $^B A_i^a$, only by terms containing derivatives of χ_a and no dynamical variable. This confirms the results of our analysis.

On the generalization to time-dependent frames.—The results above can be extended when $\partial_t \chi_a \neq 0$ by assigning a dynamical role to χ_a themselves. This can be done by adding the corresponding conjugate momenta π^a. In fact, a canonical transformation can be defined mapping $\{\omega_i^{AB}, {}^\gamma \pi_{CD}^j\}$ to $\{^B A_i^a, \chi_b, \pi_c^j, \pi^d\}$ if the following conditions stand

$$T_b^{-1a} \pi^b + \eta^{ab} \partial_i \left(\pi_b^i - \frac{1}{\gamma} \epsilon_b{}^{cd} \chi_c \pi_d^i \right)$$
$$- 2\eta^{[a|b|} \pi_b^i \chi^{c]B} A_{ci} = 0. \quad (10)$$

Hence these constraints must be added to describe properly the dynamics. From the comparison with Barros e Sa's paper [7] one recognizes that such constraints coincide with the boost ones, if the rotation constraints hold. This confirms the equivalence of the two formulations also when $\partial_t \chi^a \neq 0$.

Our procedure allows us to infer also in this case $SU(2)$ Gauss constraints, by virtue of a change of phase-space variables to connections $\tilde{A}_i^a$ (this feature does not stand in the Barros e Sa approach). However, now the momenta π^a must change properly in order to deal with a canonical transformation. We denote by $\tilde{\pi}^a$ the new expression of conjugate momenta to χ_a. In terms of this new set of variables, the expressions (10) simplify significantly, i.e.,

$$B_a = T^{-1a}{}_b \tilde{\pi}^b = 0. \quad (11)$$

Therefore, a sort of decoupling occurs between χ_a and $\tilde{A}_i^a$ variables, since Lorentz-Gauss constraints (which involve a mixing of the full set of phase-space coordinates) are equivalent to two sets of constraints, each one acting on a single couple of conjugate variables only. Furthermore, it is worth noting that χ_a behave as the lapse function and the

shift vector, since the corresponding momenta are constrained to vanish.

Loop quantum gravity in a local Lorentz frame.—Summarizing the previous analysis, the action of GR with the Holst modification can be written in a generic local Lorenz frame as follows:

$$S = \int d^4x \left[\tilde{\pi}_a^i \partial_t \tilde{A}_i^a + \tilde{\pi}^a \partial_t \chi_a - \frac{1}{\sqrt{g} g^{tt}} H + \frac{g^{ti}}{g^{tt}} H_i \right.$$
$$\left. + (^\gamma p^{cd}{}_{AB} + 2\chi^{d\gamma} p^{0c}{}_{AB})\omega_t^{AB} \epsilon^b{}_{cd} S_b^{-1a} G_a + \lambda^a B_a \right],$$
$$(12)$$

λ^a being Lagrangian multipliers.

The action (12) can be obtained directly from (2) using the definitions (5) and (8) and [12]. Once a canonical quantization is performed and constraints are implemented *à la* Dirac, conditions (11) can be easily solved in the most natural operator ordering taking wave functional not depending on χ_a variables.

Therefore, we end up with $SU(2)$ Gauss constraints only, apart from the supermomentum and super-Hamiltonian ones (which do not depend on the local Lorentz frame, since they are Lorentz scalars), and we formally infer the same formulation as in LQG with the time gauge.

The quantization can now be performed starting from the holonomy-flux algebra. In fact, since $\tilde{A}_i^a$ are connections of the $SU(2)$ group, holonomies $h(\tilde{A})_\alpha$ along curves α and momentum fluxes $\pi(S)$ across surfaces S can be defined. Their algebra is the same one as in LQG with the time gauge and this confirms that the quantization can be performed as in [13]. Therefore, the Hilbert space turns out to be a certain completition over the space of distributional connections, whose measure is the Ashtekar-Lewandowsky one, while basis vectors are invariant spin networks [14].

Within this scheme Lorentz transformations are generated by the following operators

$$R_a = G_a + \epsilon_a{}^{bc} \chi_b B_c,$$
$$K_a = B_a + \frac{1 - \sqrt{1 + \chi^2}}{\chi^2} \epsilon_a{}^{bc} G_b \chi_c. \quad (13)$$

These relations can be inverted and this implies that the validity of the Hamiltonian constraints is equivalent to the invariance under the action of the Lorentz group. Therefore, the local Lorentz symmetry, and, in particular, the boost one, is actually preserved on a quantum level.

For instance, let us consider the area operator A. Given a surface S and an edge e having one intersection in a point p (where the tangent to e does not belong to S), the action of the operator $A(S)$ on the parallel transport along e is given by

$$A(S)h_e(A) = \gamma\sqrt{1+\chi^2}\sqrt{O}h_e(A), \quad O = \frac{1}{1+\chi^2}\eta^{ab}\tau_a\tau_b,$$
$$(14)$$

τ_a being $SU(2)$ generators. The Immirzi parameter comes from the redefinition of generators we have to perform in order to reproduce the algebra of G_a.

Thus the area spectrum is discrete and this discrete structure does not depend on χ_a.

Comparison with covariant theory.—The results of previous sections seem to conflict with the formulation of Alexandrov [9]. In his works he attempted to quantize gravity by using a Lorentz connection A_i^X, X running on the whole set of Lorentz indexes. Lorentz covariance is manifest, so that the Gauss constraints of a Lorentz gauge theory are inferred. Furthermore, he removed second-class constraints by replacing Poisson brackets with Dirac ones, while here we use (5), thus solving at the same time second-class constraints and the boost ones for parametric χ_a. Hence, the two formulations can be matched by inserting these expressions of π_{ab}^i and $\omega^{ab}{}_i$ into the Gauss constraints of the Lorentz group, such that among them only three independent conditions remain. In this respect, it can be shown that connection (9) can be obtained from the one introduced in section 2 of [9] [equation (B11)] by adding some terms containing derivatives of χ_a. However, Alexandrov did not consider the transformation to the true $SU(2)$ connections (8). The crucial role of such variables can be traced back to the fact that A_i^X can be written via a Lorentz transformation as $(0, -\gamma\tilde{A}_i^a)$ [15].

Conclusions.—It has been proposed to extend LQG to a generic Lorentz frame and the corresponding Hamiltonian formulation has been provided. We distinguished the case of a time-independent frame from that with $\partial_t\chi_a \neq 0$. In the former, χ_a can be treated as parameters and a generalization of Ashetkar-Barbero-Immirzi connections (8) can be given, so finding the structure of an $SU(2)$ gauge theory. In the latter, even though the dynamical role of χ_a must be taken into account, nevertheless the full dynamical information is contained into the dependence on $\tilde{A}_i^a$, since physical states do not depend on χ_a.

In both cases, since Gauss constraints have been inferred, the quantization of gravity can be performed. We want to point out that this is an important extension of the work by Barros e Sa [7], who demonstrated that the standard quantization with the time gauge is well grounded, since the corresponding gauge condition can be safely fixed. However, he did not investigate the possibility to quantize in a moving frame or without any gauge fixing. Here, as far as we know we performed the first attempt to quantize gravity within the LQG framework taking no restriction on the local Lorentz frame.

The formulation is fully consistent with the case when the time gauge holds since the geometrical meaning of canonical variables adopted in both frameworks is the

same. As a confirmation of this statement, the action of the area operator has been evaluated in a generic Lorentz frame, finding that its spectrum is discrete and it does not depend on the observer. This quantum symmetry reflects the invariance of the classical metric tensor under Lorentz transformations. This analysis definitely clarifies how tangent space symmetries are preserved in a quantum gravity regime. Furthermore, this result can be linked with the findings of [16], which tightly constraint any sort of violation or modification of the Lorentz symmetry arising in a quantum gravity setting.

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