Generalization of the Landau-Lifshitz-Gilbert Equation for Conducting Ferromagnets

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We propose an extension of the Landau-Lifshitz-Gilbert (LLG) equation by explicitly including the role of conduction electrons in magnetization dynamics of conducting ferromagnets. The temporal and spatial dependent magnetization order parameter $\mathbf{m}(\mathbf{r}, t)$ generates both electrical and spin currents that provide dissipation of the energy and angular momentum of the processing magnet. The resulting LLG equation contains highly spatial dependence of damping term and thus micromagnetic simulations based on the standard LLG equation should be reexamined for magnetization dynamics involving narrow domain walls and spin waves with short wavelengths.

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Despite the phenomenological nature of the Landau-Lifshitz-Gilbert (LLG) equation [1], it has been widely used for interpreting and predicting vast experimental results such as domain wall structure, magnetization reversal, and magnetic noise. The powerful universal micromagnetic simulation codes [2] based on the LLG equation have already become a standard tool for studying the magnetization dynamics of complex structure. The key term in the LLG equation is the damping, which is usually written in a simplest form $\alpha_0 \mathbf{m} \times \partial_t \mathbf{m}$ [$\mathbf{m}(\mathbf{r}, t)$ is the order parameter of the magnetization and $\partial_t \mathbf{m} \equiv \partial \mathbf{m}/\partial t$]. Since many calculated magnetic properties such as magnetic hysteresis are not sensitive to the damping parameter, one has enjoyed the simplicity and usefulness of the LLG equation for a long time. However, there are other cases where the details of the damping matter. For example, the current-induced spin torque [3] directly competes with the damping and thus the switching threshold depends on the strength and forms of the damping. A number of theoretical studies have already shown that the extension of the LLG is needed in general [4–7], but it is unclear whether inclusion of more damping terms would describe the magnetization better due to complexity of many damping mechanisms. In conducting ferromagnets, the damping mechanism is clearer: the most significant source of damping is due to the conduction electron that carries away the excess angular momentum of the precessing ferromagnet via interband and intraband transitions [8,9]. In this Letter, we construct a new LLG equation by explicitly including the conduction electron-mediated damping.

The essential idea is that the spin current generated by the time-dependent magnetization becomes a damping torque, similar to the spin-pumping induced damping at ferromagnetic-nonmagnetic interfaces [10,11]. For a spatially varying magnetization, the induced spin current is nonuniform and thus one expects that the change of the spin angular momentum or the damping torque, associated with the spin current, is also spatially dependent. Indeed, we find that the resulting dynamic equation contains a damping tensor which depends on the spatial derivative to the magnetization order parameter. We will show how the generalized LLG equation alters the magnetization dynamics in several cases. In general, micromagnetic simulation based on our improved LLG equation should be developed for highly nonuniform magnetization dynamics.

Let us start with a simplest model for the interaction of the magnetization and the conduction electron spin,

$$i\hbar \frac{\partial}{\partial t}\Psi = \left[\frac{p^2}{2m_e} - J_{ex}\boldsymbol{\sigma}\cdot\mathbf{m}(\mathbf{r}, t)\right]\Psi, \tag{1}$$

where $\boldsymbol{\sigma}$ is the Pauli matrix and $\mathbf{m}(\mathbf{r}, t)$ is the unit vector $|\mathbf{m}| = 1$ representing the direction of the local magnetization. Since the order parameter $\mathbf{m}(\mathbf{r}, t)$ varies in space and time, we first rotate the quantization axis from a fixed axis ($\mathbf{e}_z$) to the axis parallel to $\mathbf{m}$ at given ($\mathbf{r}, t$), i.e., $\Psi = U(\mathbf{r}, t)\varphi$ where $U = \exp(-i\frac{\theta}{2}\boldsymbol{\sigma}\cdot\mathbf{n})$, θ is the angle of the rotation and $\mathbf{n} = \mathbf{e}_z \times \mathbf{m}/|\mathbf{e}_z \times \mathbf{m}|$ represents the axis of the rotation. After a straightforward algebra, Eq. (1) can be written in the following form,

$$i\hbar \frac{\partial}{\partial t}\varphi = \left[eV^s + \frac{(\mathbf{p} - e\mathbf{A}^s)^2}{2m_e} - J_{ex}\sigma_z\right]\varphi, \tag{2}$$

where the scalar potential is $V^s = -(i\hbar/e)U^\dagger \partial_t U$ and the vector potential $\mathbf{A}^s = (i\hbar/e)U^\dagger \nabla U$. As the magnetization vector $\mathbf{m}$ varies slowly in space and in time, we can treat the scalar and vector potentials as a perturbation in the unperturbed Hamiltonian $H_0 = p^2/2m_e - J_{ex}\sigma_z$, which describes two spin-up and spin-down bands with respect to the local magnetization direction. The scalar and vector potentials V^s and $\mathbf{A}^s$ can then be projected to these two spin bands. Specifically, the electric and magnetic fields for each band can be introduced,

$$E_i \equiv -\partial_i V^s - \partial_t A_i^s = \pm(\hbar/2e)(\partial_t \mathbf{m} \times \partial_i \mathbf{m})\cdot\mathbf{m} \tag{3}$$

and

$$B_i \equiv \epsilon^{ijk}(\partial_j A_k^s - \partial_k A_j^s) = \mp(\hbar/2e)\epsilon^{ijk}(\partial_j \mathbf{m} \times \partial_k \mathbf{m})\cdot\mathbf{m}, \tag{4}$$

where $\pm$ stands for spin-up and spin-down bands, the subscript (i, j, k) denotes the spatial coordinate (x, y, z), ϵ^{ijk} is the antisymmetric unit tensor. The above electric and magnetic fields have already been identified by a number of papers [12–15]. These electric and magnetic fields affect the electron transport properties in several significant ways. First, it had been shown that the magnetic field, Eq. (4), has a profound effect for the dynamics of a vortex wall [13]. Recently, the focus has been on the role of the electric field. As shown by Barnes and Maekawa [16] and Saslow [17], this electric field can induce an electromotive force and produce a finite voltage for a moving domain wall as long as $\partial_t \mathbf{m}$ is not parallel to $\partial_i \mathbf{m}$. Yang *et al.* included these fields in the semiclassical equation of motion to study the electron dynamics [14]. Bazaliy *et al.* [18] explicitly introduced an interaction of the applied electric current $\mathbf{j}_e$ with the vector potential $-\mathbf{j}_e \cdot \mathbf{A}^s$ as the source of the current-driven spin torque in the nonuniform ferromagnet. Duine [19] and Tserkovnyak and Wong [20] have extended this spin torque by including the spin relaxation to study the domain wall dynamics. If one includes these fields in the calculation for the conductivity, one finds a domain wall resistance due to mixing of two spin bands [21].

We intend to explicitly integrate these fields in the LLG equation. To do so, we first realize that these fields are the effective fields for the conduction electrons, not for the local magnetization $\mathbf{m}$. Thus, there is no direct interaction between the magnetic field and the local magnetization, i.e., $H' \neq -\mathbf{m} \cdot \mathbf{B}$. Instead, the electron *orbital* receives a Lorentz force. By using Ohm's law for each spin band, the electrical current j_i^e and spin current $\mathbf{j}_i^s$ can be obtained. Explicitly, the electrical current is

$$j_i^e = G^{\uparrow}(\mathbf{B})E_i^{\uparrow} + G^{\downarrow}(\mathbf{B})E_i^{\downarrow} = \frac{PG_0\hbar}{2e}(\partial_t \mathbf{m} \times \partial_i \mathbf{m}) \cdot \mathbf{m}, \quad (5)$$

where $G^{\uparrow(\downarrow)}$ is the conductivity for the spin-up (-down) band, $G_0 = G^{\uparrow} + G^{\downarrow}$, and $P = (G^{\uparrow} - G^{\downarrow})/(G^{\uparrow} + G^{\downarrow})$ is the spin polarization of the ferromagnet. In general, the conductivity depends on the magnetic field $\mathbf{B}$ due to the ordinary magnetoresistive effect. For the transition metal ferromagnet considered here, the ordinary magnetoresistance is typically very small even for a magnetic field as large as a few tesla. We thus neglect the magnetic field dependence of the conductivity, i.e., $G^{\sigma}(\mathbf{B}) = G^{\sigma}$. We point out that this electric current, Eq. (5), is precisely the electromotive force voltage V_{emf} considered by Barnes and Maekawa [16] when one defines $V_{\text{emf}} = (G^0)^{-1} \int j_e d^3x$.

Similarly, we obtain the spin current [22]

$$\mathbf{j}_i^s = \frac{g\mu_B}{2e}(G^{\uparrow}E_i^{\uparrow} - G^{\downarrow}E_i^{\downarrow})\mathbf{m} = \frac{g\mu_B\hbar G_0}{4e^2}(\partial_t \mathbf{m} \times \partial_i \mathbf{m}). \quad (6)$$

One immediately realizes that the spin current does not contain the spin polarization P, and one would question whether the above spin current is correct when P is zero. We recall that the electric and magnetic fields are obtained by projecting the scalar and vector potentials in Eq. (2) to two spin bands, i.e., we are limiting ourselves to a strong ferromagnet, or more precisely $J_{ex} \gg \hbar\omega a_0/\Delta_0$, where a_0 is a lattice constant, ω is the precessing frequency, and Δ_0 is the wall width. The transition metal ferromagnets certainly satisfy the above condition. The conductivity polarization P of a ferromagnet is determined by both J_{ex} and the spin-dependent impurity potential. Even if P is small, Eq. (6) remains valid.

Interestingly, the above induced spin current can also be understood in terms of the spin-pumping formulation [10] where the spin current flows from the ferromagnetic layer to the surrounding nonmagnetic medium as a result of the precessional motion of the ferromagnet. The spin current may be expressed as $A_r \mathbf{m} \times \partial_t \mathbf{m}$, where A_r is proportional to the conductance [10]. Now let us apply the spin pumping to two identical ferromagnetic layers in contact (separated by just one-lattice constant a_0); i.e., the left layer occupies $x \leq 0$ and the right layer occupies $x \geq a_0$. We assume that the magnetization directions of the two layers differ by a small angle. The spin current pumping from left (right) to right (left) layers is $\mathbf{j}^s_{L\to R(R\to L)} = A_r(\mathbf{m} \times \partial_t \mathbf{m})_{x=0(x=a_0)}$. Thus the net spin current across the $x = 0$ plane is the difference between the pumping spin currents from the left to the right and vice versa, i.e.,

$$\begin{aligned}\mathbf{j}_i^s &= A_r(\mathbf{m} \times \partial_t \mathbf{m})_{x=0} - A_r(\mathbf{m} \times \partial_t \mathbf{m})_{x=a_0} \\ &= -A_r a_0 \partial_i(\mathbf{m} \times \partial_t \mathbf{m}). \end{aligned} \quad (7)$$

The above equation contains both longitudinal (parallel to $\mathbf{m}$) and transverse (perpendicular to $\mathbf{m}$) spin current components, i.e., $\mathbf{j}_i^s = \mathbf{j}_{i\parallel}^s + \mathbf{j}_{i\perp}^s$, where $\mathbf{j}_{i\parallel}^s = -A_r a_0(\partial_i \mathbf{m}) \times \partial_t \mathbf{m}$ and $\mathbf{j}_{i\perp}^s = -A_r a_0 \mathbf{m} \times \partial_i \partial_t \mathbf{m}$. Notice that both vectors $\partial_i \mathbf{m}$ and $\partial_t \mathbf{m}$ are perpendicular to $\mathbf{m}$ due to $|\mathbf{m}| = 1$; therefore, $\mathbf{j}_{i\parallel}^s$ is indeed parallel to $\mathbf{m}$. The longitudinal and transverse spin currents play quite different roles in the spin transport of ferromagnets. The transverse spin current can be discarded in the strong ferromagnet while the longitudinal spin current usually decays slowly, of the order of the spin diffusion length. By only keeping the longitudinal spin current, Eq. (7) is equivalent to Eq. (6).

We now return to our central point of this Letter: the effect of the induced spin current on the magnetization dynamics. In the absence of spin relaxation, the spin current is directly related to the spin torque received from the magnetization, i.e.,

$$-\boldsymbol{\tau}_e = \partial_t \mathbf{n}_s + \sum_i \partial_i \mathbf{j}_i^s, \quad (8)$$

where $\mathbf{n}_s$ is the spin density of the conduction electrons. By using Eq. (6), we find

$$\begin{aligned}-\boldsymbol{\tau}_e = \partial_t \mathbf{n}_s + \frac{g\mu_B\hbar G_0}{4e^2}\Big[\mathbf{m}\sum_i \partial_i[\mathbf{m} \cdot (\partial_t \mathbf{m} \times \partial_i \mathbf{m})] \\ + \sum_i \partial_i \mathbf{m}[(\partial_t \mathbf{m} \times \partial_i \mathbf{m}) \cdot \mathbf{m}]\Big], \end{aligned} \quad (9)$$

where we have used the fact that $\mathbf{A} \equiv \partial_t \mathbf{m} \times \partial_i \mathbf{m}$ is parallel to $\mathbf{m}$ so that we write $\mathbf{A} = (\mathbf{A} \cdot \mathbf{m})\mathbf{m}$. The spin torque on the local magnetization is simply the opposite of the above torque. By including the above torque in the standard LLG equation, we have

$$\partial_t \mathbf{m} = -\gamma \mathbf{m} \times \mathbf{H}_{\text{eff}} + \mathbf{m} \times (\mathcal{D} \cdot \partial_t \mathbf{m}), \qquad (10)$$

where γ is the gyromagnetic ratio, H_{eff} is the total effective magnetic field, and $\mathcal{D}$ is the 3×3 differential damping tensor given by

$$\mathcal{D}_{\alpha\beta} = \alpha_0 \delta_{\alpha\beta} + \eta \sum_i (\mathbf{m} \times \partial_i \mathbf{m})_\alpha (\mathbf{m} \times \partial_i \mathbf{m})_\beta, \qquad (11)$$

where $\delta_{\alpha\beta}$ is the unit matrix element, α_0 is the original damping parameter from all other sources, M_s is the saturation magnetization, $\eta = g\mu_B \hbar G_0/(4e^2 M_s)$, and we have used the identity $\partial_i \mathbf{m} = -\mathbf{m} \times (\mathbf{m} \times \partial_i \mathbf{m})$. Note that we have discarded the first two terms on the right-hand side of Eq. (9). The first term, $\partial_t \mathbf{n}_s$, is a small renormalization of the gyromagnetic ratio γ because $\mathbf{n}_s$ is parallel to $\mathbf{m}$ to the first order approximation. The second term contributes to the longitudinal torque since it is parallel to $\mathbf{m}$. We can drop this term because we are only interested in the LLG equation, which addresses the dynamics of the transverse magnetization motion.

We notice that the enhanced damping parameter η is proportional to the conductivity; this is not surprising because the larger the conductivity, the more rapidly the angular momenta are carried away by the conduction electron. Thus our new LLG equation is more significant for materials with large conductivities. A quick estimate on the magnitude of η can be readily done for transition metals. For example, if we use $G_0 = (5\ \mu\Omega\,\text{cm})^{-1}$ and $M_s = 800$ emu/cc for Permalloy, we find $\eta = 0.5\ \text{nm}^2$. For a wavelength of $\lambda = 30$ nm for the magnetization pattern, one would expect the equivalent of the damping parameter $0.5\ \text{nm}^2(2\pi/\lambda)^2 = 0.02$; this is comparable to the damping parameter from other sources. For a vortex wall where the core magnetization varies on the scale of less than 10 nm, the damping due to the new term can be dominant.

Equations (10) and (11) are the central result of this Letter. It is interesting to compare our results with other proposed LLG forms. First, it has been suggested that the damping parameter should be taken as a tensor form to account for the anisotropic damping process [7]. In their phenomenological model, the damping is due to magnetization-lattice relaxation and each matrix element of the damping tensor is modeled by constants, independent of the spatial distribution of the magnetization. Foros *et al.* [23] considered the enhanced damping from the fluctuating spin current where they also relate the spin torque to the spin current in a similar way. However, their resulting damping term is nonanalytic and it is unclear how the LLG equation can incorporate these nonanalytic results. By explicitly taking into account the disorder and electron-electron scattering, Hankiewicz [24] recently developed a damping term which contains the form of $-\mathbf{m} \times \nabla^2 \partial_t \mathbf{m}$. Our formulation presented in Eqs. (10) and (11) is considered more general: it does not rely on the detailed electron scattering mechanism and the full tensor property is derived via the general angular momentum conservation for the precessing magnet and the itinerant electrons.

We show next that the spin torque generated by the spin current flow always reduces the magnetic energy. The rate of the energy density change is $(1/M_s)dE/dt \equiv -\mathbf{H}_{\text{eff}} \cdot \mathbf{m}$. We rewrite Eq. (10) by eliminating $\partial_t \mathbf{m}$ in the right-hand side of the equation and find

$$\frac{dE}{M_s dt} = -\gamma \alpha_0' |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 - \gamma \eta' \times \sum_i |\partial_i \mathbf{m} \cdot (\mathbf{H}_{\text{eff}} - \alpha_0 \mathbf{m} \times \mathbf{H}_{\text{eff}})|^2, \qquad (12)$$

where $\alpha_0' = \alpha_0/(1 + \alpha_0^2)$, and $\eta' = \eta/(1 + \alpha_0^2 + \alpha_0 \eta |\nabla \mathbf{m}|^2)$. Both terms are definitively negative, indicating the energy relaxation. The second term represents the energy damping due to the dissipation of the angular momentum carried away by the spin current. If one keeps only the first order terms in α_0 and η, Eq. (12) reduces to a rather simple form for the energy dissipation,

$$\frac{dE}{M_s dt} = -\gamma \alpha_0 |\mathbf{m} \times \mathbf{H}_{\text{eff}}|^2 - \gamma \eta \sum_i |\partial_i \mathbf{m} \cdot \mathbf{H}_{\text{eff}}|^2. \qquad (13)$$

The magnetic energy damped into the conduction electron is dissipated through the Joule heating. To verify this, we calculate the Joule heating density

$$Q = G^{\uparrow} E^{\uparrow 2} + G^{\downarrow} E^{\downarrow 2} = \gamma \eta \sum_i |\partial_i \mathbf{m} \cdot \mathbf{H}_{\text{eff}}|^2, \qquad (14)$$

where we have used Eq. (3) and replaced $\partial_t \mathbf{m}$ by $-\gamma \mathbf{m} \times \mathbf{H}_{\text{eff}}$ in Eq. (3). Equation (14) is precisely the second term of Eq. (13).

Our generalized LLG equations, Eqs. (10) and (11), alter many earlier predictions based on constant damping parameters. Because of dependence of the damping on the spatial distribution of magnetization vectors, the dynamics of short wavelength of spin waves or narrow domain walls will be different from that of long wavelength and wide domain walls. For example, the size of the vortex core is usually very small (order of 1 nm) and the damping of the vortex core would increase by an order of magnitude compared to the conventional LLG. The detailed investigation of magnetization dynamics requires the development of new micromagnetic simulation based on Eqs. (10) and (11). In the following, we apply them to study the dynamics of a simple domain wall.

In the original LLG model, the domain wall velocity driven by the magnetic field and by the electric current is usually modeled by the Walker's trial domain wall profile [25]. In the present case where the damping tensor $\mathcal{D}$ is spatially dependent, we find that the Walker wall profile is no longer a good solution of Eq. (10) in general. One special solution is the steady-state motion of the domain wall, i.e., $\mathbf{m} = \mathbf{m}(x - v_x t)$ and thus $\partial_t \mathbf{m} = -v_x \partial_x \mathbf{m}$, where v_x is the wall velocity. Since $\partial_t \mathbf{m}$ and $\partial_x \mathbf{m}$ are parallel, the induced current and spin current are identically zero; see Eqs. (5) and (6). Thus, the steady-state wall domain does not generate the additional damping. We also want to comment that the recent proposed domain wall motion induced voltage [16] does not always exist, since the steady-state wall motion produces neither electrical current of Eq. (5) nor the spin current Eq. (6). When the applied magnetic field exceeds a Walker breakdown field, the steady-state wall motion is no longer a solution of Eq. (10). Instead, the wall shape and widths are constantly changed during wall motion. Although there is no analytical form for the wall profile, we may still estimate the *average* wall velocity at a large external field below.

The transverse domain wall in the conventional LLG equation moves along the one-dimensional magnetic wire in an oscillatory manner for a magnetic field larger than the Walker breakdown field $H_w = \alpha_0 2\pi M_s$ [25,26]. The oscillatory component of the wall velocity can be averaged out by defining an average wall velocity v_x as

$$-2v_x H_{\rm ext} M_s \equiv \left\langle \frac{\partial E}{\partial t} \right\rangle. \qquad (15)$$

The above equation has a clear interpretation: the average rate of the energy change due to the domain wall motion is solely determined by the change of the Zeeman energy (left-hand side); this is because all other energies (exchange, anisotropy, magnetostatics) are internal energies and their changes are averaged to zero in the long time limit. The rate of the energy change can then be estimated from Eq. (13) by using the approximate expressions for the wall energy at a large magnetic field [26],

$$\langle \gamma \alpha_0 |\mathbf{m} \times \mathbf{H}_{\rm eff}|^2 \rangle = (\gamma \Delta_0 H/\alpha_0)[H - \sqrt{H^2 - H_w^2}],$$

where Δ_0 is the average wall width and

$$\int |(\partial_x \mathbf{m}) \cdot \mathbf{H}_{\rm eff}|^2 dx \approx H^2/\Delta_0.$$

The domain wall velocity is thus

$$v_x = \frac{\gamma \Delta_0 H}{2\alpha_0}\left(1 - \sqrt{1 - \frac{H_w^2}{H^2}}\right) + \frac{\gamma \eta H}{2\Delta_0}. \qquad (16)$$

The second term is the additional velocity due to the new damping term. In the case of large external field $H \gg H_w = \alpha_0 2\pi M_s$, one has

$$v_x = \gamma \Delta_0 \left(\frac{\alpha_0 \pi^2 M_s^2}{H} + \frac{\eta H}{2\Delta_0^2} \right).$$

The first term is a Walker velocity that is inversely proportional to the applied field. The second term is proportional to the field.

In summary, we have proposed a Landau-Lifshitz-Gilbert equation for conducting ferromagnets by explicitly taking into account nonuniform magnetization. The origin of this additional damping is due to the electron spin current carrying away the nonequilibrium angular momentum and energy of the ferromagnet, leading to a spatial dependent damping tensor.

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