Dynamically Error-Corrected Gates for Universal Quantum Computation

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Scalable quantum computation in realistic devices requires that precise control can be implemented efficiently in the presence of decoherence and operational errors. We propose a general constructive procedure for designing robust unitary gates on an open quantum system without encoding or measurement overhead. Our results allow for a low-level error correction strategy solely based on Hamiltonian engineering using realistic bounded-strength controls and may substantially reduce implementation requirements for fault-tolerant quantum computing architectures.

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Physical realizations of quantum information processing hold the potential to solve problems in physics simulation, combinatorial analysis, and secure communications with unprecedented power compared with known classical counterparts [1]. The discovery that arbitrarily accurate (fault-tolerant) quantum computation (QC) may be supported by real-world imperfect devices provided that the overall noise is below a certain threshold [2] indicates that no fundamental obstacle prevents this power from being harnessed in principle. The requirement on errors affecting an operation on physical qubits may be quantified in terms of an appropriate error per gate, EPG. While suggestive evidence exists that EPGs well above 10^{-3} can be tolerated through exploitation of concatenated quantum error correction (QEC) [3] and postselection [4,5], implementation requirements remain daunting. The main obstacle in generating precise unitary transformations on an open quantum system is due to the fact that a control prescription realizing a desired gate in the ideal (closed-system) limit no longer works accurately in the presence of an uncontrollable environment: typically, the resulting EPG will be proportional to both the noise strength and gating time. Our goal is to show how, for open quantum systems undergoing linear decoherence, information about the errors of a fixed set of primitive gates may be exploited to construct dynamically corrected gates (DCGs) which achieve a significantly smaller EPG using realistic bounded-strength control resources.

Our approach successfully merges elements from different techniques for high-fidelity coherent quantum control—including composite [6–8] and strongly modulating pulses [9] from nuclear magnetic resonance (NMR), as well as dynamical decoupling (DD) methods for decoherence suppression [10,11]. Common to these approaches is the idea of enforcing active error cancellation through purely Hamiltonian (open-loop) control, bypassing the need for measurement or feedback implicit in QEC. Similar to strongly modulating pulses, DCGs coherently average out unwanted evolution by cascading primitive control operations within a self-contained composite block. Unlike the standard NMR setting, however, where the unintended error component is either classical or induced by a known spin Hamiltonian, DCGs must operate without assuming complete knowledge or control over the underlying open-system Hamiltonian. As we shall see, a general analytic prescription for the required control modulation may be established by suitably incorporating known relationships between errors into control design.

Our results advance open-loop approaches to error control in several ways. While DD-inspired constructions have been employed to obtain shaped pulses which approximate ideal "δ pulses" by self-refocusing specific unwanted couplings up to some degree [12,13], existing schemes do not incorporate the effect of a generic quantum environment. In contrast, DCGs provide a complete prescription for achieving universal QC with reduced error solely based on unitary manipulations. Although procedures for combining DD with logic gates have been established in [14] in the so-called "bang-bang" limit and rigorously analyzed in [15], our approach has the key advantages of avoiding unphysical bang-bang controls from the outset, along with the need for encoding and stringent time synchronization. Thus, DCGs can be instrumental as a low-level error control strategy for bringing quantum fault tolerance closer to current capabilities.

Error and control assumptions.—Consider a target system S consisting of n qubits, coupled to an environment (or bath) B. The joint evolution is governed by a Hamiltonian of the form $H = H_S \otimes I_E + H_e$, where the error Hamiltonian $H_e = I_S \otimes H_B + H_{SB}$ is responsible for both pure bath evolution via H_B and unwanted interaction via $H_{SB} = \sum_\alpha S_\alpha \otimes B_\alpha$, for operators $S_\alpha(B_\alpha)$ acting on S (B), with S_α traceless. We focus on the case where S is driftless, $H_S = 0$, and subject to arbitrary linear decoherence:

$$H_{SB} = \sum_{i=1}^{n} \sum_{\alpha=x,y,z} S_\alpha^{(i)} \otimes B_\alpha^{(i)} = \sum_{i,\alpha} \sigma_\alpha^{(i)} \otimes B_\alpha^{(i)}. \qquad (1)$$

We further assume the bath operators H_B and $B_\alpha^{(i)}$ to be bounded [16], but otherwise *unknown*.

Control over S is implemented by applying a time-dependent Hamiltonian $H_{\rm ctrl}(t)$. Let the propagator $U_{\rm ctrl}(t_2, t_1) = \mathcal{T}_+ \exp\{-i \int_{t_1}^{t_2} H_{\rm ctrl}(t)dt\}$, in units $\hbar = 1$, effect an intended gate on S in the absence of H_e. If $H_{SB} \neq 0$, application of the same control Hamiltonian results in an actual propagator $U(t_2, t_1)$ whose action deviates from the intended one due to error dynamics induced by H_e. The EPG may be quantified in terms of a Hermitian error phase operator $\Phi(t_2, t_1)$ by writing $U(t_2, t_1) = U_{\rm ctrl}(t_2, t_1) \times \exp[-i\Phi(t_2, t_1)]$. Physically, Φ is related to the time-dependent error Hamiltonian which describes the joint evolution in the "toggling frame" that follows the control [10]. The norm of Φ bounds the fidelity loss between the intended and actual evolution of S [17]. Φ may contain pure bath terms that have no effect on the reduced system dynamics, e.g., when $H_{SB} = 0$, $\Phi(t_2, t_1) = (t_2 - t_1)H_B$. To avoid ambiguity, we define an operator A modulo pure bath terms by letting $A_{{\rm mod}B} = A - 2^{-n}{\rm Tr}_S A$.

We specify the available control resources by assuming access to the following switchable control Hamiltonians:

$$\{h_x(t)X^{(i)}, h_y(t)Y^{(i)}, h_{zz}(t)Z^{(i)}Z^{(j)}\}, \tag{2}$$

for appropriate control inputs $h_a(t)$. This allows any unitary evolution on S to be approximated within the circuit model of QC [1]. Specifically, a universal gate set is given by (i) NOOP gates, in which no operation is performed on some or all of the qubits; (ii) arbitrary single-qubit rotations on qubit i, $X^{(j)}_{2\theta} = e^{-i\theta X^{(j)}}$ and $Y^{(j)}_{2\theta} = e^{-i\theta Y^{(j)}}$; (iii) two-qubit phase gates between qubits i and j, $Z^{(ij)}_{2\theta} = e^{-i\theta Z^{(i)}Z^{(j)}}$. Realistic control profiles $h_a(t)$ will be constrained in many ways due to limited pulse-shaping capabilities. We incorporate finite-power and finite-bandwidth constraints by assuming the existence of a minimum switching time $\tau_{\rm min}$ for modulation and by requiring all control strengths $h_a(t)$ to be bounded by $h_{\rm max}$. A gate realized using a single control input in a predetermined manner shall be referred to as primitive—for instance, gates implemented by turning on and off a Hamiltonian from the above set according to a rectangular profile may be called primitive.

In control-theoretic terms, our goal is as follows: Given a desired unitary gate $U_{\rm gate}$ in the universal set, devise a control procedure $H_{\rm ctrl}(t)$ using Hamiltonians in the available repertoire, Eq. (2), such that (i) $U_{\rm ctrl}(t_2, t_1) = U_{\rm gate}$; (ii) the error $\|\Phi(t_2, t_1)_{{\rm mod}B}\|$ is significantly reduced compared with the primitive EPG. We construct an analytic perturbative solution which does not resort to measurements, extra qubit resources, or any quantitative knowledge of the error Hamiltonian H_e except its algebraic structure, Eq. (1). Our solution is perturbative in the sense that $\|\Phi(t_2, t_1)_{{\rm mod}B}\|$ becomes proportional to $\tau^2_{\rm min}$. Since naive switching of $H_{\rm ctrl}(t)$ would yield error phases that scale with $O(\tau_{\rm min}\|H_{SB}\|)$, this implies EPGs reduced by a factor of $O(\tau_{\rm min}\|H_e\|)$.

Error combination and cancellation.—The first step toward DCGs is to quantify the error phase arising from cascading N gates. Let $U = U_N U_{N-1} \cdots U_1 U_0$, where $U_0 = I_{SB}$ and the jth operation U_j during $[t_{j-1}, t_j]$, which is intended to generate $U_{{\rm ctrl},j} = U_{\rm ctrl}(t_j, t_{j-1})$ in the absence of H_e, has an EPG Φ_j. Up to the first order in $\max_j(\|\Phi_j\|)$, the total error Φ_U is

$$\Phi_U = \sum_{j=1}^{N} U^\dagger_{{\rm ctrl},j-1}\Phi_j U_{{\rm ctrl},j-1} + \Phi^{[2+]}, \tag{3}$$

where $\|\Phi^{[2+]}\| = O(N^2 \max\|\Phi_j\|^2)$. Each error Φ_j can be computed up to the first order in $(t_i - t_{i-1})\|H_{SB}\|$ as

$$\Phi^{[1]}_j = \int_{t_{j-1}}^{t_j} U^\dagger_{\rm ctrl}(t, t_{j-1})H_e U_{\rm ctrl}(t, t_{j-1})dt + \Phi^{[2+]}_j,$$

with $\|\Phi^{[2+]}_j\| = O[(t_i - t_{i-1})^2\|H_e\|^2]$. Here, $A^{[2+]}$ includes corrections of second- or higher-order powers in Φ_j to A.

The next step is to seek a combination of gates which removes the combined error (at least) up to the first order. This is reminiscent of DD approaches to QEC, whereby the time scale difference between the action of the errors (in the non-Markovian regime) and the available controls is leveraged to reduce or symmetrize the effect of the environment [10]. While several flavors of DD are possible depending on system and design specifics, the formulation relevant to our purpose is Eulerian DD (EDD) [18], which implements DD using bounded-strength controls and guarantees robustness against systematic control faults. Consider a set of unitary operators on S, which form a (projective) representation $\{U_{g_i}\}$ of the so-called DD group $\mathcal{G} = \{g_i\}_{i=1}^D$, and let Ω be the subspace of all traceless (modulo pure bath) operators which obey the following decoupling condition:

$$\left(\sum_i U^\dagger_{g_i} E U_{g_i}\right)_{{\rm mod}B} = 0, \quad \forall\ E \in \Omega.$$

A good DD group ensures, in particular, that Ω contains all operators generated by the errors $\{S_\alpha\}$ we wish to correct. For single-qubit error generators as in Eq. (1), the smallest DD group is $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$ under the n-fold product representation in terms of collective Pauli matrices $\{I^{(\rm all)}, X^{(\rm all)}, Y^{(\rm all)}, Z^{(\rm all)}\}$, with $S^{(\rm all)}_\alpha = \bigotimes_{i=1}^n S^{(i)}_\alpha$. Consider the Cayley graph associated to a set of generators $\{h_j\}_{j=1}^L$ of $\mathcal{G}$. In this directed graph, each vertex represents a group element, and two vertices g_i, g_j are connected with an edge labeled by the generator h if and only if $g_j = g_i h$. Cayley graphs always have an Eulerian cycle, that is, there exists a closed sequence of $L \times D$ connected edges that visits each edge exactly once. The Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is shown in Fig. 1(left), along with an Eulerian cycle beginning (and ending) at the identity. Consider now a sequence of primitive gates that implement the generators $\{h_j\}$ in the same order they appear in this Eulerian cycle. Provided that same generators are implemented by same gates along the path, the error of the full sequence may be obtained using Eq. (3):

$$\Phi_{\rm EDD} = \sum_{j,i} U^\dagger_{g_i}\Phi_{h_j}U_{g_i} + \Phi^{[2+]}_{\rm EDD},$$

where Φ_{h_j} is the EPG associated to h_j and $\|\Phi^{[2+]}_{\rm EDD}\| =$

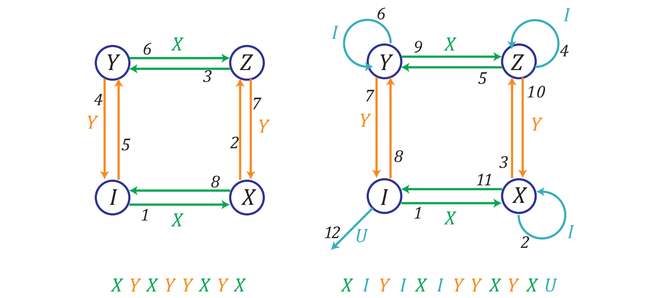

FIG. 1 (color online). Left: The Cayley graph for $\mathbb{Z}_2 \times \mathbb{Z}_2$ represented by collective Pauli operators $\{I, X, Y, Z\}$, together with an Eulerian cycle marked by numbers. Edges are labeled by the generators X, Y, and arrows denote the direction of action. Right: Modified Cayley graph supporting an Eulerian path that cancels errors in a nonidentity gate. The new edges correspond to sequences with the same (leading-order) error.

$O(\max\|\Phi_{h_j}\|^2)$. As long as $\Phi_{h_j} \in \Omega$ up to the first order, then irrespective of how each primitive gate is implemented, $(\Phi_{\rm EDD})_{{\rm mod}B} = 0$ up to the first order. Thus, a dynamically corrected NOOP can be effected with a significantly smaller error compared to the free evolution.

Dynamically corrected gates.—The above construction does not directly extend to transformations other than NOOP [19]. In order to build composite gates which (as in EDD) not only cancel the combined error but (unlike EDD) effect a nontrivial rotation U, we need to use relationships between the errors of the primitive gates. One such relationship is obtained when two combinations of gates M_I and M_U, intended to generate I and $U \neq I$, respectively, have the same error to the leading order. Explicitly: $M_U = U\exp(-i\Phi)$ and $M_I = \exp(-i\Phi)$. This relationship suggests the following modifications of the Cayley graph used for NOOP: (i) To every vertex other than identity, attach a self-directed edge labeled with M_I; (ii) To the identity vertex attach a new vertex representing U through an edge labeled by M_U, see Fig. 1 (right). This new graph possesses an Eulerian path starting at I and ending at U, which implements a DCG U. The net EPG for the corresponding gate sequence is given by

$$\Phi_{\rm DCG} = \Phi_{\rm EDD} + \sum_i U_{g_i}^\dagger \Phi U_{g_i} + \Phi_{\rm DCG}^{[2+]},$$

where $\|\Phi_{\rm DCG}^{[2+]}\| = O[\max(\|\Phi_{h_j}\|^2, \|\Phi\|^2)]$. As long as Φ_{h_j} and Φ all belong (up to the first order) to the subspace Ω of correctable errors, $(\Phi_{\rm DCG})_{{\rm mod}B} = 0$, up to $O(\|\Phi\|)$. Thus, U is implemented with a significantly smaller error compared with M_U.

Resource requirements and performance.—We now provide an explicit construction of dynamically corrected NOOP, single-, and two-qubit gates within the control scenario specified in Eq. (2). Let 2θ parametrize the rotation angle. For each operation, we assume that (i) the control profile $h_\theta(t)$ in the interval $[t_1, t_2]$ is obtained by stretching, scaling, and additions of a fixed reversible pulse shape $h_0(t)$, $t \in [0, 1]$, e.g., we let $h_\theta^{[t_1,t_2]}(t) = \theta h_0(\frac{t-t_1}{t_2-t_1})$; (ii) both positive and negative values for $h_0(t)$ are available. Rectangular pulses provide the simplest illustrative setting, with $h_0(t) = 1$ if $t \in [t_1, t_2]$, and zero otherwise. A concrete example of control sequences that yield equal errors, yet differ in the intended action is given by the following two piecewise-constant control profiles:

$$h_1(t) = (\theta/\tau)[h_0(t/\tau) - h_0(2 - t/\tau)], \tag{4}$$

$$h_2(t) = (\theta/2\tau)h_0(t/2\tau), \qquad \tau \geq \tau_{\rm min}. \tag{5}$$

In Eq. (4), $h_1(t)$ corresponds to a sequence of two primitive gates intended to implement the identity over time 2τ. In Eq. (5), $h_2(t)$ corresponds to a primitive gate of duration 2τ implementing $C_{2\theta}$. One may show [19] that for any choice of the basic shape h_0, the errors are the same up to the leading order, $\Phi_1^{[1]} = \Phi_2^{[1]}$, and belong to the subspace $\Omega_{2,\rm inhom}$ of inhomogeneous two-qubit system-bath terms spanned by $\{\sigma_\alpha^{(i)}\sigma_\beta^{(j)} \otimes B_{\alpha\beta}^{(ij)}\}$, with $\alpha \neq \beta$. The modified Cayley graph for a DCG $C_{2\theta}$ can be obtained by specializing the Eulerian path depicted in Fig. 1 to the case where U originates from $h_2(t)$ and I from $h_1(t)$, respectively. Notice that the final edge associated with $h_2(t)$ connects the identity to the new vertex representing the desired $C_{2\theta}$. The required gates $X^{(\rm all)}$ and $Y^{(\rm all)}$ are implemented through collective single-qubit Hamiltonians. Direct calculation shows that the errors for all the primitive gates used involve only single-qubit terms, $\Omega_{\rm single} \subset \Omega_{2,\rm inhom} \subset \Omega$. Thus, the combined error is zero modulo pure bath terms, up to the first order. The resulting DCG circuit appears in Fig. 2. The DCG for $C_{2\theta}$ is longer in duration than a single primitive $C_{2\theta}$ gate by a factor of 16. The error $\|(\Phi_{\rm DCG}^{[2+]})_{{\rm mod}B}\|$ on the other hand is found to be reduced by a factor of order $O(\tau\|H_e\|)$.

The above estimate indicates that significant improvement is expected for sufficiently small $\tau_{\rm min}$. Quantitative supporting results are given in Fig. 3 for an algorithm which prepares a 3-qubit cat state $|\psi_{\rm cat}\rangle$ for a spin-based qubit device coupled to a spin bath. While the simulation is performed for a relatively small system, it does show the expected behavior of DCGs in the regime where individual primitive EPGs are small. For this reason, the net fidelity loss using primitive EPG is chosen as the x axis in Fig. 3, as opposed to a less objective measure such as the coupling strength. A systematic control error is also included, al-

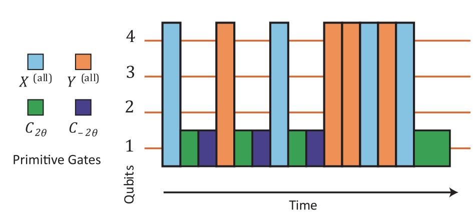

FIG. 2 (color online). Quantum circuit for a DCG on qubit 1. The NOOP is implemented by dropping all $C_{\pm 2\theta}$ gates.

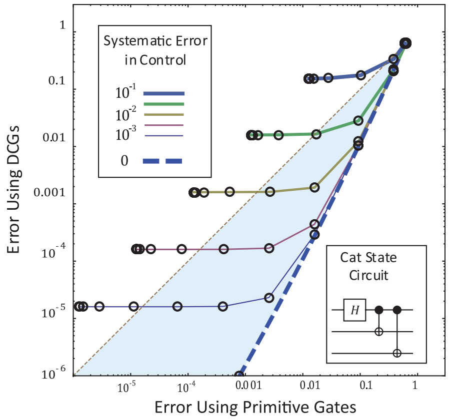

FIG. 3 (color online). DCG performance in a 3-qubit cat-state algorithmic benchmark, using the metric $1 - \mathrm{Tr}\sqrt{\rho_{\rm cat}\rho_{\rm out}\rho_{\rm cat}}$, with $\rho_{\rm cat} = |\psi_{\rm cat}\rangle\langle\psi_{\rm cat}|$ and $\rho_{\rm out}$ the actual output state. Linear decoherence from a 5-qubit bath and systematic control errors are included. Hadamard and CNOT gates appearing in the inset quantum circuit are decomposed as sequences of (2 and 6, respectively) physical primitive gates, and each such gate is replaced by a DCGs, for a total of $(2 + 2 \times 6) \times 16 = 256$ control time slots. Every bath spin I_a interacts via a Heisenberg coupling with every other system or bath spin, $H_e = \Gamma\sum_{a<b}\vec{I}^{(a)} \cdot \vec{I}^{(b)} + A\sum_{i,a}\vec{\sigma}^{(i)} \cdot \vec{I}^{(a)}$, in such a way that (in units of $\tau_{\rm min}^{-1}$) $\Gamma = 1$ and $\log_{10}A = -1, -1.4, \ldots, -5.8$, corresponding to different circles along a given curve in the figure. Rectangular pulse profiles are used throughout, systematic pulse-length errors being included by modifying $h_0(t)$ as $h_0(t)(1 + \epsilon)$. Line thickness proportionally represents error strength ϵ. Note the bunching of data points at low EPG for each ϵ, signaling a regime dominated by systematic error.

lowing its interplay with the bath-induced errors to be analyzed. Several conclusions may be drawn. For ideal control, the EPG reduction is manifest in the change of slope between dynamically corrected (thick dashed line) and uncorrected (narrow dashed line) data, resulting in the "cone of improvement" of the procedure. The smaller the primitive EPG, larger amounts of systematic control faults may be tolerated, resulting in larger regions of improvement within the above cone. Notice that while systematic error compensation along the standard Cayley graph is ensured [18], robustness need not be retained along the added arms in the modified graph. Thus, no further improvement from reducing the primitive EPG arises once uncompensated systematic error dominates over bath-induced error, leading to the observed performance plateaux.

Discussion.—We have shown how to synthesize unitary gates able to approximate ideal gates in a universal set with a quadratic error with respect to the original EPG without encoding or measurements. While our present construction addresses arbitrary linear decoherence, different algebraic error structures may be tackled by modifying the DD group. Notably, for dephasing-dominated error processes, simpler DCGs based on 6 (vs 16) primitive gates suffices, which may be relevant to recently proposed fault-tolerant superconducting architectures under biased noise [20]. Our results provide the starting point for several generalizations. The restriction to driftless systems and the pulse shape assumptions may be relaxed, at the expenses of a more complex search for identifying distinct primitive gate sequences with the same leading error [19]. Systematic control faults can be further mitigated by concatenating DCGs with composite pulses, at the expenses of longer control sequences. Conceptually, our analysis points to suggestive trade offs between available error knowledge and error correctability in open-loop strategies. Ultimately, we believe that DCGs will further boost the practical significance of dynamical error control for quantum engineering and fault-tolerant computation.

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