Implementation of the Duality between Wilson Loops and Scattering Amplitudes in QCD

Yuri Makeenko
Institute of Theoretical and Experimental Physics, Moscow, Russia

Poul Olesen
The Niels Bohr International Academy, The Niels Bohr Institute, Copenhagen, Denmark
(Received 29 October 2008; published 20 February 2009)

We generalize modern ideas about the duality between Wilson loops and scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory to large-N (or quenched) QCD. We show that the area-law behavior of asymptotically large Wilson loops is dual to the Regge-Veneziano behavior of scattering amplitudes at high energies and fixed momentum transfer, when the quark mass is small and/or the number of particles is large. We elaborate on this duality for string theory in flat space, identifying the asymptotes of the disk amplitude and the Wilson loop of large-N QCD.

DOI: 10.1103/PhysRevLett.102.071602 PACS numbers: 11.25.Tq, 12.38.Aw

This Letter is inspired by a remarkable recent discovery of the duality between Wilson loops and scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills (SYM) theory (see [1] for a review of this subject). SYM theory differs from QCD by the contents of matter fields [6 scalars and 4 spinors in the adjoint representation of the SU(N) color group, thereby providing an extended $\mathcal{N} = 4$ supersymmetry] and has attracted a significant interest over the last three decades as a toy model for certain aspects of QCD, in particular, for the relation between QCD and strings. Adding the extra fields makes the dynamics of SYM theory much simpler than that of QCD, and it enjoys the famous anti–de Sitter-space/conformal-field-theory (AdS/CFT) correspondence which, in particular, relates SYM Wilson loops to an open superstring in $\mathrm{AdS}_5 \otimes S^5$ [2]. While essential ingredients of QCD—asymptotic freedom and quark confinement—are not present in $\mathcal{N} = 4$ SYM, it captures many features of QCD perturbation theory.

The (finite part of the) 4-gluon on-shell scattering amplitude in SYM theory has the form

$$A(s,t) = A_{\mathrm{tree}} e^{f(\lambda)\log^2(s/t)} \tag{1}$$

(where s and t are usual Mandelstam's variables) as was conjectured [3] on the basis of three-loop calculations. To explain Eq. (1), the Wilson-loop–scattering-amplitude (WL-SA) duality was introduced [4] at large 't Hooft couplings λ, which has been then advocated in SYM perturbation theory [5]. This duality states that the scattering amplitude (divided by the kinematical factor A_{tree}) equals the Wilson loop for a rectangle whose vertices x_i are related to the momenta p_i of scattering gluons by

$$p_i = K(x_i - x_{i-1}), \tag{2}$$

where $K = 1/2\pi\alpha'$ is the string tension.

The function $f(\lambda)$ also appears in the anomalous dimensions of cusped Wilson loops and operators of twist two. It has been recently found as a solution to the equation [6] derived from spin chains. Its perturbative solution reproduces the three [3,7] and four [8] loop SYM results, while numerical [9] and analytical [10] solutions reproduce the $\sqrt{\lambda}$ behavior of $f(\lambda)$ for large λ originally found [11] using the AdS/CFT correspondence. The next orders in $1/\sqrt{\lambda}$ also agree with the superstring calculations [12], thereby providing a remarkable test of the AdS/CFT correspondence.

Our goal in this Letter is to find out what features of the described WL-SA duality (if any) remain valid for QCD and, in particular, how it is possible to maintain the relation of the type (2) which would relate large momenta in scattering amplitudes with loops of large size. Of course this is not possible in QCD perturbation theory, where $|p| \sim 1/|x|$ because of dimensional ground. But nonperturbatively a dimensional parameter $K \approx (400\ \mathrm{MeV})^2$ appears in QCD, which shows up in the area-law behavior of asymptotically large Wilson loops:

$$W(C) \overset{\text{large } C}{\propto} e^{-KS_{\min}(C)}, \tag{3}$$

where $S_{\min}(C)$ is the area of the minimal surface bounded by C, which results in confinement. Strictly speaking, this requires large N or the quenched approximation.

As is well-known by now, a string theory, which QCD is supposedly equivalent to, is not the simplest Nambu-Goto string. Some extra degrees of freedom living on the string are required which are most probably conveniently described by a presence of extra dimensions. The asymptotic behavior (3) is nevertheless universal for large loops. Also, there is a considerable amount of evidence from lattice gauge calculations in $2 + 1$ and $3 + 1$ dimensions for various N that the Nambu-Goto action describes the behavior of the Wilson loops quite well and the transition from perturbative to stringy behavior takes place "at surprisingly small distances" [13]. There also exists a number of other comparisons between results from the Nambu-Goto action, e.g., between the closed string spectra, and SU(N) for various N (see [14] and references therein). This action has the well-known anomaly for $d \neq 26$, which however is suppressed for long strings [15].

This remarkable success of the Nambu-Goto string in flat space as an effective action leads us to reconsider the relation between the Wilson loop $W(C)$ and the corresponding string wave functional. We then obtain scattering amplitudes in large-N (or quenched) QCD by properly summing $W(C)$ over paths and find that the WL-SA duality holds in a kinematical region of large s and fixed t when only large loops, for which the area law (3) sets in, are essential in the sum over paths. Thus obtained scattering amplitudes, quite involved in general, are of the Regge-Veneziano type when the quark mass is small and/or the number of external particles is large.

Our starting point is the standard representation of Green's functions of M colorless composite quark operators [e.g., $\bar{q}(x_i)q(x_i)$] in terms of the sum over all Wilson loops passing via the points x_i $(i = 1, \ldots, M)$, where the operators are inserted:

$$\begin{aligned} G &\equiv \left\langle \prod_{i=1}^{M} \bar{q}(x_i)q(x_i) \right\rangle_{\text{conn}} \\ &= \int_0^\infty d\mathcal{T} e^{-m\mathcal{T}} \int_0^{\mathcal{T}} d\tau_{M-1} \prod_{i=1}^{M-2} \int_0^{\tau_{i+1}} d\tau_i \\ &\quad \times \int_{\substack{z(0)=z(\mathcal{T})=x_0 \\ z(\tau_i)=x_i}} \mathcal{D}z(\tau) J[z(\tau)] W[z(\tau)]. \end{aligned} \tag{4}$$

Here the weight for the path integration is

$$J[z(\tau)] = \int \mathcal{D}k(\tau) \mathrm{sp} P e^{i\int_0^{\mathcal{T}} d\tau[\dot{z}(\tau)\cdot k(\tau) - \gamma(\tau)\cdot k(\tau)]} \tag{5}$$

for spinor quarks and scalar operators. In Eq. (4) $W(C)$ is the Wilson loop in pure Yang-Mills theory at large N (or quenched), m is the quark mass, and τ is the proper-time variable. For finite N, correlators of several Wilson loops have to be taken into account. The derivation of this formula and the references can be found in [16].

The on-shell M-particle scattering amplitudes can be obtained from the Green function (4) by the standard Lehman-Symanzik-Zimmerman reduction. When making the Fourier transformation, it is convenient to represent M momenta of the (all incoming) particles by the differences $\Delta p_i = p_{i-1} - p_i$. Then momentum conservation is automatic while an (infinite) volume V is produced, say, by integration over x_0. It is convenient to introduce a momentum-space loop $p_\mu(\tau)$ which is piecewise constant:

$$p(\tau) = p_i \quad \text{for } \tau_i < \tau < \tau_{i+1}. \tag{6}$$

Because the derivative $\dot{p}(\tau) = -\sum_i \Delta p_i \delta(\tau - \tau_i)$ with $\Delta p_i \equiv p_{i-1} - p_i$, we write in the Fourier transformation $\sum_i \Delta p_i \cdot x_i = \int d\tau p(\tau) \cdot \dot{z}(\tau)$, which is manifestly parametric invariant.

Making the Fourier transformation, we obtain

$$G(\Delta p_1, \ldots, \Delta p_M) = \int_0^\infty d\mathcal{T} e^{-m\mathcal{T}} \int_0^{\mathcal{T}} d\tau_{M-1} \prod_{i=1}^{M-2} \int_0^{\tau_{i+1}} d\tau_i \int_{z(0)=z(\mathcal{T})=0} \mathcal{D}z(\tau) e^{i\int_0^{\mathcal{T}} d\tau \dot{z}(\tau)\cdot p(\tau)} J[z(\tau)] W[z(\tau)], \tag{7}$$

where $p(\tau)$ is piecewise constant as in Eq. (6). We do not integrate over $z(0) = z(\mathcal{T})$, which would produce the (infinite) volume factor because of translational invariance.

To calculate the scattering amplitudes, we have to substitute the area-law behavior (3) of asymptotically large Wilson loops into Eq. (7) and to integrate over the paths. In general, this would lead us to very complicated integrals, but the calculation drastically simplifies if we use the representation of the minimal area as a boundary functional that was introduced by Douglas [17] in his celebrated solution of the Plateau problem. We shall use one of the equivalent forms of the Douglas functional:

$$\begin{aligned} A[\sigma] = &-\frac{1}{4\pi} \int_0^{\mathcal{T}} d\tau_1 d\tau_2 \dot{x}(\tau_1) \cdot \dot{x}(\tau_2) \\ &\times \ln(1 - \cos\{2\pi[\sigma(\tau_1) - \sigma(\tau_2)]/\mathcal{T}\}), \end{aligned} \tag{8}$$

where $0 < \sigma(\tau) < \mathcal{T}$ is a reparametrization $[\sigma'(\tau) \geq 0]$. The functional (8) is to be minimized with respect to $\sigma(\tau)$ with the minimizing function $\sigma_*(\tau)$ being, of course, contour-dependent. Then $A[\sigma_*]$ is equal to the minimal area $S_{\min}(C)$, while in general $A[\sigma] \geq A[\sigma_*] = S_{\min}(C)$.

In fact (8) is well-known as the classical boundary action in string theory. It appears for the tree-level disk amplitude with Dirichlet boundary conditions in the Polyakov string formulation after integrating over the string fluctuations inside the disk, i.e., over $X(r, \theta)$ with $r < 1$, $0 \leq \theta < 2\pi$, and fixing the value $X(1, \theta) \equiv x(\theta)$ at the boundary. The appearance of the function $\sigma_*(\theta)$ is related to a subtlety associated with fixing conformal gauge [18]. The decoupling of the Liouville field is possible only in the interior of the disk, while its boundary value determines the function $\sigma_*(\theta)$ at the classical level. The path integral over the boundary value of the Liouville field then restores the invariance under reparametrizations of the boundary in quantum theory.

Motivated by this fact, Polyakov [19] proposed to identify the Wilson loop in large-N QCD with the tree-level string disk amplitude integrated over reparametrizations of the boundary contour. It is convenient to conformally map the disk into the upper half-plane, so the disk boundary is mapped into the real axis parametrized by $t(\tau) = \tan(\pi\tau/\mathcal{T})$, $-\infty < t < +\infty$. Then we write

$$\begin{aligned} W(C) = \int \mathcal{D}s(t) \exp\bigg[&\frac{K}{2\pi} \int_{-\infty}^{+\infty} dt_1 dt_2 \dot{x}(t_1) \cdot \dot{x}(t_2) \\ &\times \ln|s(t_1) - s(t_2)| \bigg], \end{aligned} \tag{9}$$

where the path integral over $s(t)$ [with $s'(t) \geq 0$] restores the invariance under reparametrizations.

In spite of the fact that the right-hand side of Eq. (9) is derivable for a bosonic string in $d = 26$ or superstring in

$d = 10$, we shall use it only for asymptotically large loops or, equivalently, very large K, when the integral over reparametrizations has a saddle point at $s(t) = s_*(t)$. This is crucial for reproducing Eq. (3).

It is easy to calculate a (reparametrization-invariant) functional Fourier transformation

$$W[p(\cdot)] = \int \mathcal{D}x e^{i\int p\cdot dx} W[x(\cdot)] \tag{10}$$

of the disk amplitude (9) for piecewise constant $p(t)$. Substituting (9) into Eq. (10) and performing the Gaussian integration, we get

$$W[p(\cdot)] = \int \mathcal{D}s(t) \exp\left[\alpha' \int_{-\infty}^{+\infty} dt_1 dt_2 \dot{p}(t_1)\cdot \dot{p}(t_2) \ln|s(t_1) - s(t_2)|\right], \tag{11}$$

which is of the same form as (9) only with K replaced by $1/K = 2\pi\alpha'$.

Since $p(t) = p_j$ at the jth interval for the stepwise discretization, the only effect of the reparametrization is to change the values of t_j's for s_j's, keeping their cyclic order. This is a discrete version of the reparametrization transformation. Note that the stepwise discretization of $x(t)$ itself is not possible since it would violate the continuity of the world line of the string end.

The stepwise discretization (6) naturally results in the M-particle (off-shell) Koba-Nielsen amplitudes which are invariant under the SL(2; $\mathbb{R}$) projective transformation $s \Rightarrow (as + b)/(cs + d)$ with $ad - bc = 1$ because the projective group is a subgroup of reparametrization transformations. To derive them, we first note that

$$\int_{-\infty}^{+\infty} dt_1 dt_2 \dot{p}(t_1)\cdot \dot{p}(t_2) \ln|s(t_1) - s(t_2)| = -\frac{1}{2}\int_{-\infty}^{+\infty} \frac{ds_1 ds_2}{(s_1 - s_2)^2}\{p[t(s_1)] - p[t(s_2)]\}^2 \tag{12}$$

for the integral in the exponent in (11). The integration over s_1 or s_2 on the right-hand side has divergences when s_1 and s_2 lie on adjacent sides $k = l \pm 1$. If we omit the sides with $k = l \pm 1$, then the integrations over s_1 and s_2 are perfectly finite, resulting in

$$\frac{1}{2}\sum_{k\neq l\pm 1} \int_{s_{k-1}}^{s_k} ds_1 \int_{s_{l-1}}^{s_l} ds_2 \frac{(p_k - p_l)^2}{(s_1 - s_2)^2} = \sum_{k\neq l\pm 1} \Delta p_k \cdot \Delta p_l \log|s_k - s_l| + \sum_j \Delta p_j^2 \log\frac{(s_j - s_{j-1})(s_{j+1} - s_j)}{(s_{j+1} - s_{j-1})}, \tag{13}$$

which is projective invariant.

Choosing the measure to be

$$\mathcal{D}s = \prod_i ds_i \frac{(s_{i+1} - s_{i-1})}{(s_i - s_{i-1})(s_{i+1} - s_i)}, \tag{14}$$

which is also invariant under the projective transformation, we arrive at

$$W(\Delta p_1, \ldots, \Delta p_M) = \int_{s_{j-1}<s_j} \prod_i ds_i \prod_{k\neq l} |s_k - s_l|^{\alpha'\Delta p_k \Delta p_l} \times \prod_j \left(\frac{(s_j - s_{j-1})(s_{j+1} - s_j)}{(s_{j+1} - s_{j-1})}\right)^{\alpha'\Delta p_j^2 - 1}, \tag{15}$$

where the integration over s_j emerges from the path integral over reparametrizations in Eq. (11). This is known as the Lovelace choice [20] (see [21]), which reproduces some projective-invariant off-shell string amplitudes known since the late 1960s. The more familiar on-shell tachyon amplitudes can be obtained from Eq. (15) by setting $\alpha'\Delta p_j^2 = 1$.

Fixing in Eq. (15) the remaining SL(2; $\mathbb{R}$) invariance in the standard way, we obtain the scalar amplitudes in the Koba-Nielsen variables. For the case of 4 scalars this reproduces the Veneziano amplitude

$$A(\Delta p_1, \Delta p_2, \Delta p_3, \Delta p_4) = \int_0^1 dx x^{-\alpha(s)-1}(1 - x)^{-\alpha(t)-1}, \tag{16}$$

where $\alpha(s) = \alpha' s + 1$ and $s = -(\Delta p_1 + \Delta p_2)^2$, $t = -(\Delta p_2 + \Delta p_3)^2$ are usual Mandelstam's variables (for Euclidean metric). Here the tachyonic condition $\alpha'\Delta p_j^2 = 1$ has not to be imposed. While Eq. (15) results in $\alpha(0) = 1$, an arbitrary value of the intercept $\alpha(0)$ can be reached by properly changing the measure (14).

We are now in a position to perform the main task of this Letter: to substitute the area-law behavior (3) of $W(C)$ into the path integral (7) and to find out for what momenta the asymptotically large loops dominate. As we shall see, typical momenta will be large for large loops. As is already explained, we substitute Eq. (9) for Eq. (3), which gives the same for large loops (or large $K = 1/2\pi\alpha'$).

Interchanging the order of integration over $z(\tau)$ and $\sigma(\tau)$ [or $s(t)$] and easily doing a Gaussian path integral, we obtain

$$G(\Delta p_1, \ldots, \Delta p_M) = \int_0^\infty d\mathcal{T} e^{-m\mathcal{T}} \int_0^{\mathcal{T}} d\tau_{M-1} \prod_{i=1}^{M-2} \int_0^{\tau_{i+1}} d\tau_i \int \mathcal{D}\sigma(\tau) \times \int \mathcal{D}k(\tau) \mathrm{sp} P e^{\alpha'/2 \int_0^{\mathcal{T}} d\tau_1 \int_0^{\mathcal{T}} d\tau_2 [\dot{k}(\tau_1)+\dot{p}(\tau_1)]\cdot[\dot{k}(\tau_2)+\dot{p}(\tau_2)] \ln(1-\cos\{(2\pi)/(\mathcal{T})[\sigma(\tau_1)-\sigma(\tau_2)]\}) - i\int_0^{\mathcal{T}} d\tau \gamma(\tau)\cdot k(\tau)}. \tag{17}$$

This expression is rather close to the disk amplitude (11), except for the additional integration over k. But for the case where m is small (or M is very large), the integral over $\mathcal{T}$ in Eq. (17) is dominated by $\mathcal{T} \sim (M-1)/m$ which is large for m small. This is because $\prod_i^{M-1} d\tau_i \sim \mathcal{T}^{M-1}$. Noting that typical values of $k \sim 1/\mathcal{T}$ are essential in the path integral over k for large $\mathcal{T}$, we can disregard $k(\tau)$ in the exponent in Eq. (17) so the integral over k factorizes. Making the change of the variables from σ to s, we finally obtain from Eq. (17) the product of the momentum-space disk amplitude (11) times factors which do not depend on p. Therefore, Eq. (17) exactly reproduces for piecewise constant $p(\tau)$ the (off-shell) Koba-Nielsen amplitude (15) as $m \to 0$.

It still remains to discuss in what kinematical region of momenta Δp_i our derivation is legible; that is, only asymptotically large loops are essential in the path integral over $z(\tau)$ in Eq. (7). A physical intuition suggests, from the spectrum of a classical string, this should be the case at least for asymptotically large s and large $t \lesssim s$. This indeed agrees with our formulas, where the value of $\alpha(0)$ is not essential in this region. But the domain of applicability of our approach is broader and extends to large negative values of t. However, when $-t \ll s$ becomes large enough, there are no longer reasons to expect the contribution of large loops to dominate over perturbation theory, which comes from integration over small loops in Eq. (7). Therefore, our formula for the 4-point scattering amplitude is valid only for asymptotically large s and fixed t ($|t| \gtrsim 1/\alpha'$), associated with small angle or fixed momentum transfer. The tachyon issue, which is a short distance phenomenon [15], is then irrelevant. For smaller values of $|t| \lesssim 1/\alpha'$ the results become sensitive to the choice of the measure in the ansatz (9).

As distinct from previous approaches to reggeization in perturbative QCD, in particular, from that based on the evolution equation [22] for Regge trajectories, our approach deals with large loops usually associated with nonperturbative effects. Actually we are dealing with the quark-antiquark Regge trajectory, whose QCD calculation was pioneered in [23], rather than with the Pomeron.

When m is not small and/or M is not large, one should consider the full expression (17). We can split there the k integral into two domains with small and large $\dot{k}$. Then the former will appear as a Regge-Veneziano behaved factor coupled to the rest of the integrand.

Thus, in conclusion we see that the area-law behavior of Wilson loops is dual to the Regge-Veneziano behavior of scattering amplitudes at high energies and small angles, when quark mass is small and/or the number of produced particles is large, but this ceases to be valid when the momentum transfer is large. This is how the exponential falloff of the 4-particle amplitude with large $-t \sim s$, which is unavoidable in string theory [24], does not happen in our consideration.

We thank R. Marotta for pointing out that the amplitude (15) is projective invariant.

[1] L. F. Alday and R. Roiban, Phys. Rep. **468**, 153 (2008).
[2] J. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998); S.-J. Rey and J. Yee, Eur. Phys. J. C **22**, 379 (2001).
[3] Z. Bern, L. J. Dixon, and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005).
[4] L. F. Alday and J. Maldacena, J. High Energy Phys. 06 (2007) 064.
[5] J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B795**, 385 (2008); A. Brandhuber, P. Heslop, and G. Travaglini, Nucl. Phys. **B794**, 231 (2008); J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B795**, 52 (2008).
[6] B. Eden and M. Staudacher, J. Stat. Mech. (2006) P11014; N. Beisert, B. Eden, and M. Staudacher, J. Stat. Mech. (2007) P01021.
[7] A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko, and V. N. Velizhanin, Phys. Lett. B **595**, 521 (2004).
[8] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, Phys. Rev. D **75**, 085010 (2007); F. Cachazo, M. Spradlin, and A. Volovich, Phys. Rev. D **75**, 105011 (2007).
[9] M. K. Benna, S. Benvenuti, I. R. Klebanov, and A. Scardicchio, Phys. Rev. Lett. **98**, 131603 (2007).
[10] A. V. Kotikov and L. N. Lipatov, Nucl. Phys. **B769**, 217 (2007); B. Basso, G. P. Korchemsky, and J. Kotanski, Phys. Rev. Lett. **100**, 091601 (2008).
[11] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. **B636**, 99 (2002).
[12] S. Frolov and A. A. Tseytlin, J. High Energy Phys. 06 (2002) 007; R. Roiban and A. A. Tseytlin, J. High Energy Phys. 11 (2007) 016; Phys. Rev. D **77**, 066006 (2008).
[13] M. Lüscher and P. Weisz, J. High Energy Phys. 07 (2002) 049.
[14] A. Athenodorou, B. Bringoltz, and M. Teper, Phys. Lett. B **656**, 132 (2007).
[15] P. Olesen, Phys. Lett. B **160**, 144 (1985).
[16] Y. Makeenko, *Methods of Contemporary Gauge Theory* (Cambridge University Press, Cambridge, England, 2002), p. 249.
[17] J. Douglas, Trans. Am. Math. Soc. **33**, 263 (1931).
[18] O. Alvarez, Nucl. Phys. **B216**, 125 (1983); B. Durhuus, P. Olesen, and J. L. Petersen, Nucl. Phys. **B232**, 291 (1984); A. G. Cohen, G. W. Moore, P. C. Nelson, and J. Polchinski, Nucl. Phys. **B267**, 143 (1986); A. M. Polyakov, *Gauge Fields and Strings* (Harwood, Academic, Chur, Switzerland, 1987), p. 183.
[19] A. M. Polyakov (unpublished); P. Orland, Nucl. Phys. **B605**, 64 (2001); V. S. Rychkov, J. High Energy Phys. 12 (2002) 068.
[20] C. Lovelace, Phys. Lett. B **32**, 490 (1970).
[21] P. Di Vecchia, in *String Quantum Gravity and Physics at the Planck Energy Scale, Erice 1992* (World Scientific, Singapore, 1993), p. 16; A. Liccardo, F. Pezzella, and R. Marotta, Mod. Phys. Lett. A **14**, 799 (1999).
[22] I. A. Korchemskaya and G. P. Korchemsky, Phys. Lett. B **387**, 346 (1996).
[23] R. Kirschner and L. N. Lipatov, Phys. Rev. D **26**, 1202 (1982); Nucl. Phys. **B213**, 122 (1983).
[24] D. Gross and P. Mende, Phys. Lett. B **197**, 129 (1987).