Clustering Properties and Model Wave Functions for Non-Abelian Fractional Quantum Hall Quasielectrons

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We present model wave functions for quasielectron (as opposed to quasihole) excitations of the unitary Z_k parafermion sequence (Laughlin, Moore-Read, or Read-Rezayi) of fractional quantum Hall states. We uniquely define these states through two generalized clustering conditions: they vanish when either a cluster of $k+2$ electrons is put together or when two clusters of $k+1$ electrons are formed at different positions. For Abelian fractional quantum Hall states ($k=1$), our construction reproduces the Jain quasielectron wave function and elucidates the difference between the Jain and Laughlin quasielectrons.

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The connection between conformal field theory (CFT) and fractional quantum Hall (FQH) states [1,2] provides model wave functions for non-Abelian ground states and their quasihole excitations. A central result of the CFT-FQH connection has been the prediction that the addition of several units of flux creates *multiple* degenerate pinned quasihole states which exhibit non-Abelian statistics. In particular, some of the Read-Rezayi (RR) [3] series of non-Abelian states are thought to be experimentally relevant to the $\nu=5/2$ and $\nu=12/5$ FQH plateaus.

Despite these successes, the FQH-CFT connection has failed to produce *unique* model wave functions for the Laughlin quasielectron states. This is due to the fact that, until recently, previous attempts at introducing quasielectrons invariably necessitated using antiholomorphic coordinates $z^\star$ (in some form) and then projecting to the lowest Landau level (LLL); this procedure can be done in several ways, leading to different polynomial wave functions. For the non-Abelian states, quasielectron wave functions are not known. Recently, several authors [4] have succeeded in expressing the Jain model quasielectron wave functions for the Laughlin hierarchy sequence as CFT correlators. However, several Abelian quasielectron models exist (due to Laughlin, Jain, Girvin, Halperin, and others), and the fundamental *physical* differences between them are not understood.

In this Letter we provide an explicit construction of LLL quasielectron model wave functions for the Z_k RR sequence. The RR Z_k FQH ground states are uniquely defined as the smallest degree polynomials that vanish when $k+1$ particles cluster together. Our purpose is to find similar *physical* clustering conditions (Hamiltonians) that uniquely define the one-quasielectron state. Since quasielectrons involve the removal of flux, and hence the lowering of the total degree of the polynomial wave function, a one-quasielectron wave function of the RR states can no longer vanish when $k+1$ particles come together. We find two kinds of quasielectrons: an Abelian one-quasielectron wave function of the RR Z_k sequence vanishes when $2k+1$ particles come together and when two clusters, each of $k+1$ particles, are formed at different positions. A non-Abelian one-quasielectron wave function vanishes when $k+2$ particles come together and when two clusters, each of $k+1$ particles, are formed at different positions. For $k=1$ (Laughlin states), the two clustering conditions are equivalent; our one-quasielectron states turn out to be identical to Jain's. The clustering conditions they satisfy explain the numerically observed energetic superiority of Jain's quasielectrons over Laughlin's.

We represent a partition λ with length $\ell_\lambda \le N$ as a (bosonic) occupation-number configuration $n(\lambda)=\{n_m(\lambda), m=0,1,2,\ldots\}$ of each of the LLL orbitals $\phi_m(z)=(2\pi m!2^m)^{-1/2}z^m\exp(-|z|^2/4)$ with angular momentum $L_z=m\hbar$ (see Fig. 1), where, for $m>0$, $n_m(\lambda)$ is the multiplicity of m in λ. It is useful to identify the "dominance rule" [5] (a partial ordering of partitions $\lambda>\mu$) with the "squeezing rule" [6] that connects configurations $n(\lambda)\rightarrow n(\mu)$: "squeezing" is a two-particle operation that moves a particle from orbital m_1 to m_1' and another from m_2 to m_2', where $m_1<m_1'\le m_2'<m_2$, and $m_1+m_2=m_1'+m_2'$; $\lambda>\mu$ if $n(\mu)$ can be derived from $n(\lambda)$ by a sequence of "squeezings" (see Fig. 1). An interacting LLL polynomial P_λ indexed by a *root partition* λ is defined as exhibiting a dominance property if it can be expanded in occupation-number noninteracting states (monomials) of orbital occupations $n(\mu)$ obtained by squeezing on the root occupation $n(\lambda)$:

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} v_{\lambda\mu} m_\mu. \tag{1}$$

The $v_{\lambda\mu}$ are rational number coefficients. Partitions λ can be classified by λ_1, their largest part. When any P_λ is expanded in monomials m_μ, no orbital with $m>\lambda_1$ is occupied. P_λ can be interpreted as states on a sphere surrounding a monopole with charge $N_\Phi=\lambda_1$ [7]. Uniform (ground) states on the sphere satisfy the condi-

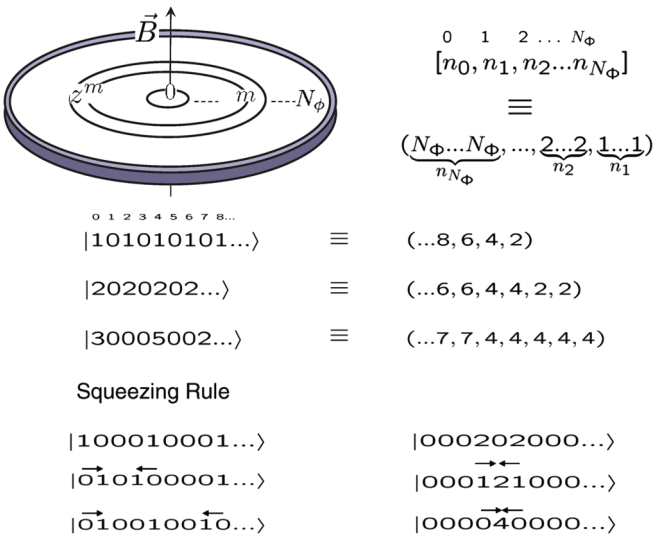

FIG. 1 (color online). Landau problem on a disk. Orbital occupation to monomial basis conversion and squeezing rule examples.

tions $L^+\psi = 0$ [highest weight (HW)] and $L^-\psi = 0$ [lowest weight (LW)] where $L^+ = E_0$, and $L^- = N_\Phi Z - E_2$, where $Z \equiv \sum_i z_i$, and $E_n = \sum_i z_i^n \partial/\partial z_i$. In a previous Letter [8], we proved, by using the HW and LW conditions, that the Jack polynomials (Jacks) of root occupation $n(\lambda^0(k,2)) = [k0k0k\ldots k0k]$ and Jack parameter $\alpha_{k,r} = -(k+1)/(r-1)$ are the ground-state wave functions of the RR Z_k sequence. The RR quasihole wave functions are also Jacks of root occupation numbers satisfying a $(k, 2)$ of a more general (k, r) Pauli principle which allows no more than k particles in r consecutive angular momentum orbitals. For the Jacks, the coefficients $v_{\lambda\mu}$ are explicitly known by recursion [5]. In our construction, we require the squeezing rule be satisfied also for the quasielectron states.

Quasielectron states satisfy only the HW condition $L^+\psi = 0$ and should represent a small local perturbation of the otherwise featureless ground-state density. We now present the root occupation $n(\lambda)$ and a set of clustering conditions which uniquely define the quasielectron wave functions. We start with the Abelian one-quasielectron added to the $\nu = \frac{k}{2}$ Jacks $J^{-(k+1)}_{\lambda^0_{k,2}}(z_1, \ldots, z_N)$ (RR ground states) of root occupation $n(\lambda^0_{k,2}) = [k0k0k0k\ldots k0k]$. By analogy with the Abelian quasihole, this should be a state of total angular momentum $L = N/2$. We add three fluxes to the ground state and obtain the occupation number $n = [000k0k0k\ldots k0k]$. The Abelian one-quasielectron state is obtained by adding $2k$ particles in the zeroth orbital (north pole): $n(\lambda^0_{k,1\ \mathrm{qp}}) = [2k00k0k0k\ldots k0k]$. Simple counting gives us $N_\Phi = \frac{2}{k}(N-k) - 1$, the correct flux for an N particle $\nu = \frac{k}{2}$ Read-Rezayi state with an Abelian one-quasielectron. Away from the north pole, the quasielectron root occupation relaxes to the bulk sequence $[k0k\ldots k0k]$.

The root occupation and the HW condition do not define the one-quasielectron polynomial wave function uniquely. We now search for a way to uniquely define the polynomial. In a previous paper [9] we showed that the HW condition on the Jacks gives an infinite set of Jacks at $\alpha = -(k+1)$ of occupations $n(\lambda^0_{k,2,s}) = [n_0 0^{s+1} k0k0k0k\ldots]$ with $n_0 = (k+1)(s+1) - 1$, and $s \geq 0$ a positive integer. For $s = 0$ these are the RR FQH ground states. For $s \geq 1$, we have $n_0 > k$ and hence these configurations contain an excess of charge at the north pole, and heal in the bulk to the RR ground-state configurations. However, as the Abelian s-quasielectron state in the Z_k sequence should have $N_\Phi = \frac{2}{k}(N-k) - s$, the orbital occupation $n(\lambda^0_{k,2,s})$ contains too much charge at the north pole. To obtain the correct N_Φ, we must ''subtract'' s particles from the zero orbital of the occupation sequence $n(\lambda^0_{k,2,s}) = [n_0 0^{s+1} k0k0k0k\ldots]$ of the Jacks given in [9], to obtain the root occupation configuration $n(\lambda^0_{k,s\ \mathrm{qp}}) = [k(s+1)0^{s+1}k0k0k\ldots k0k]$. At the explicit, first quantized wave function level, this ''subtraction'' can be done by symmetrization and padding of the Jack polynomial $J^{\alpha_{k,r}}_{\lambda^0_{k,2,s}}$ [10], but a simpler expression will be presented shortly. Defined in this way, the s-quasielectron state shares a clustering property with $J^{\alpha_{k,2}}_{\lambda^0_{k,2,s}}$ that we obtained in [9]: it vanishes when $s+1$ clusters of $k+1$ same-position particles are formed. Being HW states dominated by $n(\lambda^0_{k,s\ \mathrm{qp}})$, they also vanish when $k(s+1)+1$ particles come together at the same point as the $s+2$'s power of the difference between coordinates [9]. The angular momentum of the Abelian s-quasielectron configurations above is $l(\lambda^0_{k,s\ \mathrm{qp}}) = L_z(\lambda^0_{k,s\ \mathrm{qp}}) = \frac{s}{2}N$. The above root configurations define the *maximum* angular momentum Abelian s-quasielectron states (bunched up at the north pole) of the Laughlin, Read-Moore, and Read-Rezayi sequence. Hence, our HW Abelian ($s =$) one-quasielectron state is uniquely defined as the smallest degree polynomial satisfying the clustering conditions:

$$P(\underbrace{z_1 \ldots z_1}_{k+1}, \underbrace{z_2 \ldots z_2}_{k+1}, z_{2k+3}, z_{2k+4}, \ldots, z_N) = 0$$
$$P(\underbrace{z_1 \ldots z_1}_{2k}, z_{2k+1}, z_{2k+2}, \ldots, z_N) \sim \prod_{i=2k+1}^{N} (z_1 - z_i)^3. \quad (2)$$

For $N_\Phi = \frac{2}{k}(N-k) - 1$, the counting developed in [9] gives exactly $N+1$ linearly independent polynomials satisfying Eq. (2). They correspond to the different l_z's of the $l = \frac{N}{2}$ multiplet of states. The HW state $(l, l_z) = (\frac{N}{2}, \frac{N}{2})$ satisfies a more stringent clustering condition than Eq. (2):

$$P(\underbrace{z_1, \ldots z_1}_{2k}, z_{2k+1}, z_{2k+2}, \ldots, z_N) = \prod_{i=2}^{N} (z_1 - z_i)^3 \times J^{-(k+1)}_{\lambda^0(k,2)}(z_{2k+1}, \ldots, z_N),$$

where $n(\lambda^0(k,2)) = [k0k0k \ldots k0k]$ and $J^{-(k+1)}_{\lambda^0(k,2)}(z_{2k+1},\ldots,z_N)$ is the RR Z_k ground state for $N-k$ particles. An alternate definition which also uniquely fixes the HW one-quasielectron state is requiring that it satisfies HW, dominance, and the first clustering condition in Eq. (2). The second clustering condition in Eq. (2) is then automatically obeyed.

We now obtain explicit first quantized expressions of our states. For the Laughlin, $(k,r)=(1,2)$, $\nu=1/2$, state we find that the one-quasielectron HW wave function $P_{\lambda^0(1,1)}$ involves one symmetrization over a Jack found in [9]

$$P_{\lambda^0_{1,1\ \mathrm{qp}}}(z_1,\ldots,z_N) = \mathrm{Sym} J^{-2}_{\lambda^0_{1,2,2}}(z_1,z_1,z_2,z_3,\ldots,z_N). \quad (3)$$

Model HW wave functions for the s-qp state of *maximum* angular momentum $l = s\frac{N}{2}$ are obtained by further symmetrization over the Jacks of [9]: $P_{\lambda_{1,s\ \mathrm{qp}}}(z_1,\ldots,z_N) = \mathrm{Sym} J^{-2}_{\lambda^0_{1,2,s+1}}(z_1,z_1,z_2,z_2,\ldots,z_s,z_s,z_{s+1},z_{s+2},\ldots,z_N)$. For $k>1$, similar expressions can be obtained [10]. However, we found that our wave functions can be written in compact form using an operator first introduced by Jain [11]:

$$O(\partial_1,\ldots,\partial_N,z_1,\ldots,z_N) = \mathrm{Det}\begin{pmatrix} \partial_1 & \partial_2 & \ldots & \partial_N \\ 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ \vdots & \vdots & & \vdots \\ z_1^{N-2} & z_2^{N-2} & \ldots & z_N^{N-2} \end{pmatrix},$$

where Det denotes the determinant. We find our HW Abelian one-quasielectron states of the RR Z_k sequence, as defined by symmetrization over Jacks, are identical to

$$P_{\lambda^0_{k,1\ \mathrm{qp}}}(z_1,\ldots,z_N) = \frac{1}{\Delta} O J^{-(k+1)}_{\lambda^0_{k,2}}(z_1,\ldots,z_N), \quad (4)$$

where $\Delta = \prod_{i<j}^N (z_i - z_j)$ is the Vandermonde determinant and $J^{-(k+1)}_{\lambda^0_{k,2}}(z_1,\ldots,z_N)$ is the Jack polynomial FQH ground state of the RR Z_k sequence [8]. The right-hand side of Eq. (4) is a symmetric polynomial as the determinant operator O is antisymmetric in the z_i's. We have checked that $P_{\lambda^0_{k,1\ \mathrm{qp}}}$ in Eq. (4) exhibits a dominance property Eq. (1) with the root occupation $n(\lambda^0_{k,1\ \mathrm{qp}}) = [2k00k0k0k0k\ldots k0k]$, and satisfies the clustering conditions in Eq. (2). The $l_z = -N/2 \ldots N/2$ multiplet can be obtained by successively applying the L^- operator on $P_{\lambda^0_{k,1\ \mathrm{qp}}}$. These states also satisfy the clustering conditions in Eq. (2). The density profiles for the Read-Moore $\nu=1$ and the Read-Rezayi $\nu=\frac{3}{2}$ quasielectron are plotted in Fig. 2. For $k=1$ Laughlin states, by Eq. (4) our quasielectron wave functions can be seen to be identical to Jain's [11]. Our definition of the quasielectron through the clustering conditions Eq. (2) provides a physical explanation for the numerical finding [12] that Jain's quasielectron has a lower energy than Laughlin's [13]. We found that Laughlin's original quasielectron wave function [13] satisfies the second of the clustering conditions in Eq. (2) but not the first one. We have checked that Jain's quasielectron has a lower energy than Laughlin's due to the fact that it satisfies one extra clustering condition.

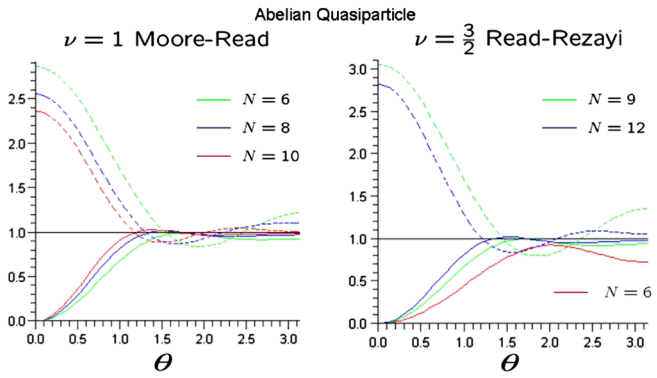

FIG. 2 (color online). Exact HW Abelian one-quasielectron (at the north pole) (dashed lines) density profiles, in units of $kN/4\pi(N-k)l^2$, for the Moore-Read ($k=2$) and Read-Rezayi $k=3$ N-particle states on the sphere. The exact LW Abelian one-quasihole density profiles (solid lines) are also plotted for reference.

Thus far we have focused on the bosonic ($m=0$) Z_k FQH states. For integer $m \geq 1$, the Read-Rezayi sequence at filling $\nu = k/(km+2)$ has the wave function $\Psi^m_{\mathrm{RR}} = \prod_{i<j}(z_i - z_j)^m J^{\alpha_{k,2}}_{\lambda^0_{k,2}}$. The HW quasielectron wave function is $\psi^{k,1\ \mathrm{qp}}_{\nu=(k/km+2)}(z_1,\ldots,z_N) = \prod_{i<j=1}^N (z_i - z_j)^m P_{\lambda^0_{k,1\ \mathrm{qp}}}(z_1,\ldots,z_N)$. The above construction of the quasielectron trivially generalizes to the entire (k,r) Jack sequence of FQH states introduced in [8].

We now construct the non-Abelian quasielectron states for the RR Z_k sequence. A non-Abelian fractionalized quasielectron will always be accompanied by a non-Abelian fractionalized quasihole, and will be composed of an electron bound to a fractionalized quasihole. As the Abelian quasihole has angular momentum $l=\frac{N}{2}$, each fractionalized non-Abelian quasihole (and fractionalized quasiparticle) has $l=\frac{N}{2k}$. The basic neutral excitation of the system is a fractionalized one-quasielectron one-quasihole state at the same flux as the FQH RR ground state $N_\Phi = \frac{2}{k}(N-k)$. As a fractionalized quasielectron and quasihole are distinguishable particles, angular momentum addition gives multiplets of states $l \equiv \frac{N}{2k} \oplus \frac{N}{2k} = \frac{N}{k}, \frac{N}{k}-1, \frac{N}{k}-2, \ldots, 2, 1, 0$ (the $l=1$ state will be missing). The HW $l=\frac{N}{k}$ state corresponds to completely separating the fractionalized quasielectron at the north pole from the fractionalized quasihole at the south pole. It is uniquely defined by the dominated polynomial of root occupation $n(\lambda^0_{k,1\ \mathrm{qp\text{-}1\ qh}}) = [k+10k-11k-11k-1\ldots1k-1]$, satisfying the clustering conditions

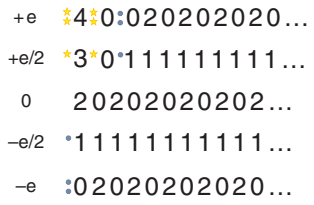

FIG. 3 (color online). Root occupation numbers for the highest weight bosonic Moore-Read ground state, Abelian (charge $-e$) quasihole, fractionalized quasihole (charge $-e/2$), Abelian (charge e) quasielectron, and fractionalized (charge $e/2$) quasielectron. The fractionalized quasielectron is a composite particle containing one electron, denoted here by two yellow star symbols, and a fractionalized quasihole (one blue circle).

$$P(\underbrace{z_1,\ldots,z_1}_{k+1},\underbrace{z_2,\ldots,z_2}_{k+1},z_{2k+3},z_{2k+4},\ldots,N)=0,$$
$$P(\underbrace{z_1,\ldots,z_1}_{k+2},z_{k+3},z_{k+4},\ldots,z_N)=0. \quad (5)$$

The HW ($l_z = l$) states of the $l = \frac{N}{k} - 1, \ldots, 2, 0$ multiplets can be uniquely defined by imposing HW, along with first clustering condition in Eq. (5), on dominated polynomials with root occupations:

$$l=\frac{N}{k};\qquad [k+10k-11k-11\ldots1k-11k-1];$$
$$l=\frac{N}{k}-1;\qquad [k+10k-11k-11\ldots1k-10k];$$
$$l=2;\qquad [k+10k-10k0k\ldots k0k0k];$$
$$l=0;\qquad [k0k0k0k\ldots k0k0k]. \quad (6)$$

The second clustering condition in Eq. (5) is then automatically obeyed. Successive application of the L^- operator yields the $l_z = l, \ldots, -l$ wave functions, which also obey the clustering conditions in Eq. (5). We can in fact prove that our states satisfy a stronger clustering condition than in Eq. (5):

$$P(\underbrace{z_1\ldots z_1}_{k+1},\underbrace{z_2\ldots z_2}_{k},z_{2k+2},z_{2k+3},\ldots,z_N)$$
$$\sim (z_1-z_2)^{2k+1}\prod_{i=2k+2}^{N}(z_2-z_i)^2(z_1-z_i)^2. \quad (7)$$

The root occupation numbers for the Moore-Read ground state and its quasiparticle excitations are shown in Fig. 3. Their density profiles are plotted in Fig. 4.

Just as in the Abelian case, there are several ways to define the non-Abelian one-quasielectron one-quasihole states, which lead to the same result. Requiring HW, dominance with respect to the root occupations Eq. (6) and the first of the clustering conditions in Eq. (5) uniquely defines the states. The second clustering condition in Eq. (5) is then automatically satisfied. Alternatively, Eq. (7) and the second clustering in Eq. (5) also uniquely define the Hilbert space of one-quasielectron one-quasihole states, although in this case further angular momentum projection

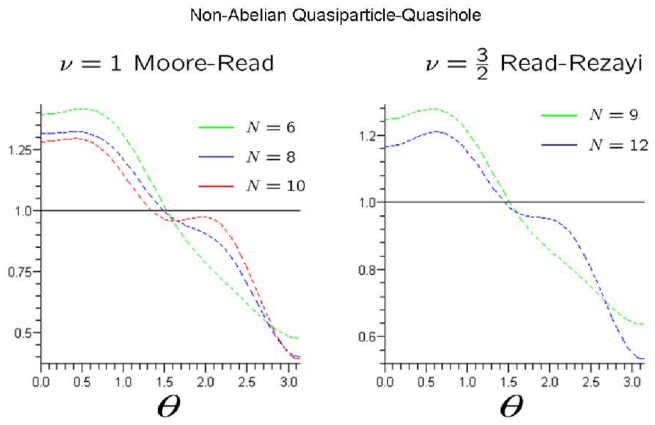

FIG. 4 (color online). Exact HW non-Abelian quasielectron-quasihole density profiles, in units of $kN/4\pi(N-k)l^2$, for the Moore-Read ($k = 2$) and Read-Rezayi ($k = 3$) N-particle states on the sphere. The fractionalized quasielectron is at the north pole while the fractionalized quasihole is at the south pole. In the thermodynamic limit, the region in the middle of the sphere at density one will dominate the density function.

is needed to obtain $\vec{L}$ eigenstates. For $k = 1$, z_2 is not different from $z_3, \ldots, z_N$, and the non-Abelian clustering conditions become identical to the Abelian ones (the Laughlin states support only Abelian excitations). We can "energetically" justify our quasielectron-quasihole wave functions. As they cannot vanish when $k + 1$ particles come together (this condition defines the RR Z_k ground-state and pure quasiholes), the lowest "energy" configuration that one can create is to require the wave function vanish in a $k + 2$ particle cluster.

In this Letter we have generalized the clustering conditions that define the RR FQH ground states and quasiholes to include the Abelian and non-Abelian quasielectron excitations. For the Laughlin state, the Jack one-quasielectron excitations are identical to Jain's.

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