Holographic Quantum Liquid

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Quantum liquids are characterized by the distinctive properties such as the low-temperature behavior of heat capacity and the spectrum of low-energy quasiparticle excitations. In particular, at low temperature, Fermi liquids exhibit the zero sound, predicted by Landau in 1957 and subsequently observed in liquid He-3. In this Letter, we ask whether such characteristic behavior is present in theories with a holographically dual description. We consider a class of gauge theories with fundamental matter fields whose holographic dual in the appropriate limit is given in terms of the Dirac-Born-Infeld action in anti–de Sitter space. We find that these systems also exhibit a sound mode at zero temperature despite having a non-Fermi-liquid behavior of the specific heat. These properties suggest that holography identifies a new type of quantum liquid which potentially could be experimentally realized in strongly correlated systems.

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Introduction.—The gauge-gravity duality [1–3] has become a useful tool for investigating strongly coupled field theories. In the class of models where this tool can be applied, the strong coupling limit of the field theory is mapped into the weak coupling, classical limit of a gravity theory, which can be studied either analytically or with minimal computer power. For example, a cousin of QCD—the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory—can be studied using this method. Such studies have pointed to a universal value of the viscosity/entropy density ratio in a wide class of strongly coupled theories (for a review, see Ref. [4]). Somewhat surprisingly, the viscosity/entropy density ratio of the quark-gluon plasma created at the Relativistic Heavy Ion Collider seems to be close to this value, indicating that gauge/gravity duality may be useful for studies of QCD.

Here we would like to see what the gauge/gravity duality has to say about strongly coupled quantum liquids. By quantum liquids we mean translationally invariant systems at zero (or low) temperature and at finite density. Given the important role that quantum liquids play in condensed matter physics, it is natural to ask whether the newly developed technique of gauge/gravity duality can give us any insights into their behavior.

The cornerstones of our understanding of quantum liquids are two phenomenological theories. These are Landau's Fermi-liquid theory [5–9] and the theory of quantum Bose liquids [7,8]. These two theories describe two different behaviors of a quantum liquid at low momenta and temperatures. In a Bose liquid, the only low-energy elementary excitation is the superfluid phonon with a linear dispersion. This leads to a T^3 behavior of the specific heat at low temperatures. The Fermi liquid has a richer spectrum of elementary excitations, consisting of fermionic quasiparticles and a bosonic branch, which contains, in particular, the zero sound. The fermions dominate the specific heat, which scales as T at low T.

In this Letter, we found, through the gauge/gravity duality, a new type of quantum liquid. The quantum liquid we consider has a T^6 behavior ($\sim T^{2p}$ in p spatial dimensions) of the specific heat at low temperature. Despite the non-Fermi-liquid behavior of the specific heat, the system supports a sound mode at zero temperature, which we will call "zero-temperature sound." The mode is almost identical to the zero sound in Fermi liquids: not only the real part of its dispersion curve is linear in momentum ($\omega = vq$), but the imaginary part has the same q^2 dependence predicted by Landau a long time ago for quantum attenuation of the zero sound [6]. (Experiments measuring the zero sound quantum attenuation are described in Ref. [10].) The difference is that in our case the zero-temperature sound velocity coincides with the first-sound velocity, while in the case of a Fermi liquid the two velocities are, in general, not equal to each other.

A specific example considered in this Letter is the $\mathcal{N} = 4$ SU(N_c) supersymmetric Yang-Mills (SYM) theory with N_f massless $\mathcal{N} = 2$ hypermultiplet fields. This theory has been suggested as a model which approximates QCD better than the theory without fundamental quarks. A string-theoretic description of this system is given by a low-energy limit of the D3/D7 brane configuration. The theory has been studied at finite temperature and density using the gauge/gravity duality [11–19]. Nevertheless, two striking aspects of this and similar systems (characterized by the Dirac-Born-Infeld action in anti–de Sitter space)—the unusual behavior of the low-temperature specific heat and the existence of the zero-temperature sound—have so far eluded attention. Their description is the main purpose and the main result of the Letter.

Preliminaries.—The $\mathcal{N}=4$ SYM theory contains fields in the adjoint representation of the gauge group only. Fields in the fundamental representation can be introduced by using the following construction [20]. In type IIB string theory, one considers a system of N_c D3-branes aligned along the (x^0, x^1, x^2, x^3) directions, and N_f D7-branes aligned along $(x^0, x^1, \ldots, x^7)$ directions in flat ten-dimensional space. In the limit of large number of colors ($N_c \gg 1$) and large 't Hooft coupling ($g_{YM}^2 N_c \gg 1$), the D3-branes are replaced by the near-horizon $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry [1], while the N_f D7-branes can be treated as probes embedded into this geometry as long as $N_f/N_c \ll 1$, so that their backreaction on the geometry can be neglected [20]. The near-horizon D3 brane metric is

$$ds^2 = \frac{r^2}{R^2}(-f_T dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}(f_T^{-1} dr^2 + r^2 d\Omega_5^2), \quad (1)$$

where R is the curvature radius of the AdS_5 (which is set to 1 in the following), and $f_T = 1 - r_H^4/r^4$. The horizon is located at $r = r_H$.

We will focus on the case where the $\mathcal{N}=2$ hypermultiplets are massless. In the dual gravity picture this is described by a "horizon-crossing" D7-branes embedding in which the distance between the D7 branes and the horizon in the $x_8 - x_9$ direction is zero [13]. The action for the D7-branes is the Dirac-Born-Infeld (DBI) action

$$S_{\mathrm{DBI}} = -N_f T_{\mathrm{D7}} \int d^8\xi \sqrt{-\det(g_{ab} + 2\pi\alpha' F_{ab})}, \quad (2)$$

where T_{D7} is the D7-brane tension, ξ_a are world volume coordinates, g_{ab} is the induced worldvolume metric and F_{ab} is the world volume $U(1)$ gauge field, which couples, at the boundary, to the $U(1)_B$ baryon number current J^μ. (The exact form of the $U(1)_B$ current operator is given in [11].)

The construction above is specific to the D3–D7 system, but we can be more general and consider Dq probe branes whose world volume include an AdS_{p+2} factor. For probe branes corresponding to massless flavors the embedding is independent of the internal directions. One example with $p = 2$ would be the defect D5 on $\mathrm{AdS}_4 \times \mathrm{S}^2$ in $\mathrm{AdS}_5 \times \mathrm{S}^5$ [21]. The AdS part of the general metric is as in Eq. (1) with $f_T = 1 - (r_H/r)^{p+1}$, where the horizon radius is related to the temperature of the black hole by $r_H = 4\pi T/(p+1)$ (with $R = 1$).

We now introduce a finite chemical potential into the system. This corresponds to turning on a nontrivial background world volume gauge field $A_0(r)$ in the bulk [11]. The DBI action becomes, in the $A_r = 0$ gauge,

$$S_{\mathrm{DBI}} = -\mathcal{N}_q V_p \int dr r^p \sqrt{1 - A_0'^2}, \quad (3)$$

where the factor $2\pi\alpha'$ is absorbed into A_0, V_p is the spatial volume of the boundary gauge theory, and $\mathcal{N}_q$ is proportional to the tension of the N_f Dq-branes [13]. For the D3–D7 system, $\mathcal{N}_7 = \lambda N_f N_c/(2\pi)^4$, as determined by the gauge/gravity duality dictionary [11]. At finite temperature, the lower limit of integration in (3) is r_H.

The solution to the embedding problem is given by

$$A_0' = \frac{d}{\sqrt{r^{2p} + d^2}}, \quad (4)$$

where $d \equiv (2\pi\alpha' \mathcal{N}_q)^{-1}\rho$ is proportional to the baryon number density ρ [13]. In all subsequent formulas, the results for the D3–D7 case are trivially recovered by setting $p = 3$ and using the relation $\alpha'^{-1} = \sqrt{\lambda}$.

Low-temperature limit of the specific heat.—One interesting hint to the nature of the phase of matter described by the probe branes with finite chemical potential is the behavior of the specific heat at low temperature. The on-shell value of the total action (which is the sum of the DBI action (3) describing fundamental degrees of freedom and the bulk gravitational action dual to the adjoint sector of the theory) directly gives us minus the thermodynamic potential Ω in the grand canonical ensemble. Substituting the solution (4) for A_0' into the action, we can write $\Omega = \Omega_{\mathrm{ad}} + \Omega_{\mathrm{fun}}$, where $\Omega_{\mathrm{ad}} \sim T^{p+1}$ is the contribution of the adjoint degrees of freedom (for the D3/D7 system $\Omega_{\mathrm{ad}} = -\pi^2 N_c^2 T^4/8$ is the free energy of $\mathcal{N}=4$ SYM theory at strong coupling), and

$$\Omega_{\mathrm{fun}} = \mathcal{N}_q V_p \int_{r_H}^{\Lambda} dr \frac{r^{2p}}{\sqrt{r^{2p} + d^2}} - \frac{\mathcal{N}_q}{p+1} \times \int d^p x \sqrt{-h(\Lambda)}, \quad (5)$$

where Λ is the ultraviolet cutoff. In Eq. (5), the local counterterm action built from the metric $h_{\mu\nu}$ induced on the slice $r = \Lambda$ by the ambient metric (1) has been added in the spirit of the holographic renormalization [22]. In the grand canonical ensemble, the potential Ω is a function of T and μ, where μ is the baryon number chemical potential related to the density and temperature via the condition $\mu = \int_{r_H}^{\infty} dr A_0'$. The integrals for Ω_{fun} and μ can be expressed in terms of the Gauss hypergeometric function:

$$\Omega_{\mathrm{fun}} = \Omega_0 - \frac{\mathcal{N}_q V_p r_H^{2p+1}}{(2p+1)d} {}_2F_1\left(\frac{1}{2}, 1 + \frac{1}{2p}; 2 + \frac{1}{2p}; -\frac{r_H^{2p}}{d^2}\right) + \frac{\mathcal{N}_q V_p r_H^{p+1}}{2(p+1)}, \quad (6)$$

$$\mu = \mu_0 - r_H {}_2F_1\left(\frac{1}{2}, \frac{1}{2p}; 1 + \frac{1}{2p}; -\frac{r_H^{2p}}{d^2}\right), \quad (7)$$

where Ω_0 and μ_0 are the zero-temperature values,

$$\mu_0 = \alpha(p) d^{1/p}, \qquad \Omega_0 = -\frac{\mathcal{N}_q V_p}{(p+1)\alpha^p} \mu_0^{p+1}, \quad (8)$$

and $\alpha(p) = B(\frac{1}{2} - \frac{1}{2p}, 1 + \frac{1}{2p})/2$. We note that the last term in Eq. (6), $\mathcal{N}_q V_p r_H^{p+1}/2(p+1) \equiv \Omega_{\rm c.t.}$, is independent of the matter density and has the same temperature dependence as the free energy of adjoint fields $\Omega_{\rm ad}$. We shall focus on the density-dependent part of the thermodynamic potential $\Delta\Omega \equiv \Omega_{\rm fun} - \Omega_{\rm c.t.}$. Equations (6) and (7) determine $\Delta\Omega$ as a function of temperature and chemical potential. At low temperature, both equations can be treated as series expansions in $T/\mu_0 \ll 1$. The baryon number density is proportional to d,

$$\rho = -\frac{1}{V_p}\frac{\partial \Omega_{\rm fun}}{\partial \mu} = \mathcal{N}_q d. \tag{9}$$

One then computes the entropy density $s(\mu, T)$ in the grand canonical ensemble

$$s(T, \mu) = -\frac{1}{V_p}\left(\frac{\partial \Delta\Omega(T, \mu)}{\partial T}\right)_{\mu, V_p}. \tag{10}$$

Using Eq. (7), we find the entropy density as a function of temperature and charge density

$$s(T, d) = s_0 + \mathcal{N}_q\left(\frac{4\pi}{p+1}\right)^{2p+1}\frac{T^{2p}}{2d}[1 + O(Td^{-(1/p)})], \tag{11}$$

where $s_0 = 4\pi\rho/[(2\pi\alpha')(p+1)]$ is the entropy density at zero temperature. This entropy is related to the fermion thermal mass (free energy), which is negative and proportional to T. Finally, the specific heat (heat capacity per unit volume) c_V at constant density is determined by $c_V = T(\partial s/\partial T)_\rho$. At low temperature ($T \ll \mu_0$) the density-dependent part of the specific heat [23] is proportional to T^{2p}:

$$c_V = \mathcal{N}_q p\left(\frac{4\pi}{p+1}\right)^{2p+1}\frac{T^{2p}}{d}[1 + O(Td^{-(1/p)})]. \tag{12}$$

This has to be contrasted with a gas of free bosons whose low temperature specific heat is proportional to T^p (a sphere of volume T^p of occupied states in momentum space, each with energy T) or a gas of fermions, whose low-temperature specific heat scales as T for any p (a shell of thickness T of occupied states above the Fermi surface contributing an energy T each). The behavior of the specific heat in Eq. (12) is suggestive of a new type of quantum liquid.

Zero-temperature sound.—The zero-temperature sound mode would manifest itself as a pole of the zero-temperature retarded flavor current density correlator [7–9]. In the dual gravity language, the pole arises as the quasinormal frequency of the background geometry [24–26]. Generically, the quasinormal spectrum is determined by fluctuations of all background fields including the metric. However, in the particular case we are dealing with, it is sufficient to consider fluctuations of the DBI U(1) field in the gravitational background (1) with the nontrivial background component A_0. Moreover, since the dual quantum field theory is isotropic, we can choose the fluctuations to depend on time, radial coordinate and one of the spatial coordinates (e.g., x_p) only

$$A_\mu(r) \to A_\mu(r) + a_\mu(r, x_0, x_p). \tag{13}$$

Substituting (13) into the DBI action (3) and expanding to second order in fluctuations, we find that the longitudinal (a_0, a_p) and transverse fluctuations do not mix. The action for the longitudinal fluctuations is

$$S^{(2)} = \frac{\mathcal{N}_q}{2}\int d^{p+1}x dr r^p \left\{\frac{(\partial_r a_0 - \partial_0 a_r)^2}{(1 - A_0'^2)^{3/2}} + \frac{(\partial_0 a_p - \partial_p a_0)^2}{r^4\sqrt{1 - A_0'^2}} - \frac{(\partial_r a_p - \partial_p a_r)^2}{\sqrt{1 - A_0'^2}}\right\}. \tag{14}$$

Making a Fourier transform,

$$a_\mu(r, x_0, x_p) = \int \frac{d\omega dq}{(2\pi)^2} e^{-i\omega x_0 + iqx_p} a_\mu(r, \omega, q), \tag{15}$$

introducing a new radial coordinate $z = 1/r$, and using the radial gauge $a_r = 0$, the equations for small fluctuations can be written as

$$\partial_z(f^3 z^{2-p} a_0') - f z^{2-p}(\omega q a_p + q^2 a_0) = 0, \tag{16a}$$
$$\partial_z(f z^{2-p} a_p') + f z^{2-p}(\omega q a_0 + \omega^2 a_p) = 0, \tag{16b}$$
$$f^2 \omega a_0' + q a_p' = 0, \tag{16c}$$

where $f(z) = \sqrt{1 + d^2 z^{2p}}$. Following the approach of [26], we use Eqs. (16) to derive the equation for the gauge-invariant variable $E = \omega a_p + q a_0$:

$$E'' + \left[\frac{(3q^2 - \omega^2 f^2)f'}{(q^2 - \omega^2 f^2)f} - \frac{p-2}{z}\right]E' + \left(\omega^2 - \frac{q^2}{f^2}\right)E = 0. \tag{17}$$

The horizon, $z = \infty$, is an *irregular* singular point of the differential Eq. (17). The solution in the vicinity of $z = \infty$ is $E(z) \sim e^{\pm i\omega z}/z$. The incoming wave boundary condition at the horizon [25] singles out one of the exponents

$$E(z) = C\frac{e^{i\omega z}}{z}(1 + O(1/z)), \tag{18}$$

where C is a constant. For $\omega z \ll 1$ we have

$$E(z) = Cz^{-1} + i\omega C. \tag{19}$$

On the other hand, for $\omega z \ll 1$ and $qz \ll 1$ with ω/q fixed, the last term on the left-hand side of Eq. (17) can be dropped, and the solution is given in terms of the Gauss hypergeometric function

$$E(z) = C_1 + C_2 z^{p-1}\left[\frac{q^2}{pf(z)} + \frac{(q^2 - p\omega^2)}{p(p-1)} \times {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \frac{1}{2p}; \frac{3}{2} - \frac{1}{2p}; -d^2 z^{2p}\right)\right]. \tag{20}$$

For $z \to \infty$ we find

$$E(z) \to C_1 + C_2(az^{-1} + b) + O(z^{-2}), \tag{21}$$

with the coefficients a and b given by

$$a = \frac{\omega^2}{d}, \qquad b = \frac{\mu_0(q^2 - p\omega^2)}{pd}, \tag{22}$$

where μ_0 is defined in Eq. (8). Matching to the expansion (19), we find the coefficients C_1 and C_2:

$$C_1 = \left(i\omega - \frac{b}{a}\right)C, \qquad C_2 = \frac{C}{a}. \tag{23}$$

The lowest quasinormal frequency is found by imposing the Dirichlet condition at the boundary, $E(0) = 0$ [26]. This condition gives the equation $C_1 = 0$, whose solution at small ω and q determines the dispersion relation of the lowest quasinormal mode,

$$\omega = \pm\frac{q}{\sqrt{p}} - \frac{iq^2}{2p\mu_0} + O(q^3). \tag{24}$$

What is the nature of this excitation? First, one can exclude the possibility that it is a superfluid phonon. Indeed, our background does not break the particle number symmetry; hence, the ground state is not a superfluid. Furthermore, the superfluid phonon width has a low-momentum behavior different from q^2, namely, q^5 in 3 spatial dimensions [7] and q^{p+2} in p spatial dimensions. The q^2 behavior of the imaginary part is characteristic of the zero sound quantum attenuation; thus, we call this mode the zero-temperature sound. Yet in other respects (such as the specific heat temperature dependence) the system does not show Fermi-liquid behavior. It is notable that the zero-temperature sound velocity in our system coincides with the velocity of the finite-temperature first sound, while in a weakly coupled Fermi liquid it is $\sqrt{p}$ times larger than the first-sound velocity.

Conclusion.—In this Letter, we have considered a general theory described by a DBI action in AdS space. We found that by turning on a chemical potential one arrives at a new type of quantum liquid. The specific heat c_V has an unusual non-Fermi-liquid T^{2p} behavior (T^6 in 3 + 1 dimensions and T^4 in 2 + 1 dimensions). The low-energy spectrum contains a gapless mode with a dispersion relation similar to the zero sound in Fermi liquids. One can speculate that the mode observed here is what the Fermi-liquid zero sound becomes when the interaction is infinitely strong. In this connection, we note that in a simple model of the Fermi liquid, the velocities of the zero and first sounds approach each other in the limit where the interaction strength (parametrized by the Fermi-liquid parameter F_0) is infinite [9].

The systems described here are strongly coupled, as they have gravity duals. It would be interesting to investigate the properties of the ground state and the zero-temperature sound in the weak-coupling regime of the $\mathcal{N} = 4$ SYM theory with $\mathcal{N} = 2$ matter hypermultiplets. We leave this problem for future work.

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