Fermi Surfaces in General Codimension and a New Controlled Nontrivial Fixed Point

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The energy of a d-dimensional Fermi system typically varies only along $d_c = 1$ ("radial") dimensions. We consider $d_c = 1 + \varepsilon$ and study a transition to superconductivity in an ε expansion. The nontrivial fixed point describes a scale invariant theory with an effective space-time dimension $D = d_c + 1$. Remarkably, the results can be reproduced by the Hertz-Millis action for the superconducting order parameter in higher effective space-time dimensions. We consider possible realizations of the transition at $\varepsilon = 1$, which corresponds to a linear Fermi surface in $d = 3$.

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The ground state of a fluid of noninteracting fermions in spatial dimension $d \geq 2$ is typically a Landau Fermi liquid with a sharp Fermi surface and quasiparticles. The Fermi liquid state can be understood within a fermionic renormalization group (RG) framework [1,2] in terms of a fixed point obtained as one focuses on modes within a bandwidth Λ of the Fermi surface and systematically reduces Λ. Unlike the BCS transition, transitions like the Stoner ferromagnetic transition that occur at finite coupling are not described by traditional flows within this approach. An alternate approach of Hertz [3], extended by Millis [4], performs a renormalization group analysis on the bosonic action for the order parameter obtained by integrating out the fermions and obtains an action nonanalytic in frequency and momentum and which yields results in agreement with those of Moriya [5] who used the self-consistent renormalization approach. However, there are reasons [6,7] to believe that integrating out the gapless fermionic modes can lead to singularity structure not evident in the Hertz-Millis analysis.

When the noninteracting band structure consists of Fermi points (as in graphene) the low energy description is a massless "relativistic" Dirac theory. In dimension $d \geq 2$ weak short-ranged interactions are irrelevant, though gap inducing phase transitions can occur at strong coupling. Without the complications of an extended Fermi surface these can be analyzed within conventional field theoretic framework for critical phenomena.

We consider a general class of problems that fall in between these two cases. The low energy theory of a system of noninteracting fermions in d spatial dimension may be characterized by the codimension d_c of the surface in momentum space where the energy gap vanishes. The ordinary Fermi surface has codimension $d_c = 1$ while the case of Fermi points has codimension d. We study the general case of codimension d_c with a eye toward gaining insight into $d_c = 2$ in $d = 3$ which corresponds to fermions with line nodes in three dimensions which arise in unconventional three dimensional superconductors and the tight binding model on the diamond lattice at half filling [8].

Fermionic RG methods of Refs. [1,2] show that whenever $d_c > 1$ all short-range interactions are irrelevant. With increasing interaction strength phase transitions which gap out the fermions are again possible. For $d_c < d$ these occur in the presence of an extended gapless "Fermi surface" and thus share the complications of the usual case $d_c = 1$. As a concrete example we study a pairing phase transition associated with superconductivity. We show that this can be accessed within the fermionic RG through a controlled ϵ expansion in the codimension $d_c 1 + \varepsilon$, yielding a nontrivial scaling structure with effective space-time dimensionality $d_c + 1$. The transition can also be analyzed within the Moriya-Hertz-Millis approach which works with a bosonic order parameter that lives in the full d space dimensions. Despite the difference in dimensionalities both approaches give identical results. Comparison of the two approaches provides valuable insight into the nature of quantum criticality in fermionic systems with an extended gapless Fermi surface.

Consider then a "generalized Fermi surface" of dimension $d - d_c$. Coordinates on this Fermi surface play the role of internal symmetry degrees of freedom and will be referred to as isospin directions. The d_c directions along which the energy grows linearly with momentum will be called the Dirac directions.

We begin in $d = 2$ when we have a Fermi line (one isospin direction, parametrized by the angle θ) and one radial Dirac direction measured by $k = K - K_F$. In this world the momentum transfer $\mathbf{q}$ is two dimensional. We are interested in $\mathbf{q}$ because any bosonic order parameter bilinear in fermions will have $\mathbf{q}$ and ω at its arguments. Let us now increase the overall dimensions by ε so that we have one isospin direction, $1 + \varepsilon$ Dirac dimensions, and $2 + \varepsilon$ directions for $\mathbf{q}$. Throughout, the Fermi surface is one dimensional. Along with ω we have a total of $D = 2 + \varepsilon$ dimensions for the $(\omega, \mathbf{k})$ vector (on which the fermion fields depend) and $D = 3 + \varepsilon$ dimensions for

the $(\omega, \mathbf{q})$ vector on which the order parameter depends. The case $\varepsilon = 1$ corresponds to our target, the Fermi line in $d = 3$ with two Dirac dimensions.

The action for this theory (with Fermi velocity $v_F = 1$)

$$S_0 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{|\mathbf{k}|<\Lambda} d^{1+\varepsilon}\mathbf{k}$$
$$\times \int d\theta \, \bar{\psi}(\omega\mathbf{k}\theta)[i\omega - k]\psi(\omega\mathbf{k}\theta) \qquad (1)$$

is invariant under the RG transformation:

$$\Lambda \to s^{-1}\Lambda \qquad \omega' = s\omega \qquad \mathbf{k}' = s\mathbf{k} \qquad (2)$$

$$\psi(\omega\mathbf{k}\theta) = s^{(3+\varepsilon)/2}\psi'(\omega'\mathbf{k}'\theta). \qquad (3)$$

The angle θ that parametrizes the Fermi surface does not renormalize. In an actual problem the Fermi line may not be a circle. The universal answers calculated here as well as the kinematical implications in the limit $\Lambda/K_F \to 0$ are insensitive to this.

Let us now add to this the BCS interaction. Recall from Refs. [1,2] that in the limit $\Lambda/K_f \to 0$ the requirement that all four lines lie in the bandwidth and obey momentum conservation means the only possible four-Fermi interactions are those with either nearly forward scattering described by a function $u(\theta_1, \theta_2)$ [which will become $u(\theta_1 - \theta_2)$ for a circular Fermi surface] and those with nearly zero incoming momentum described by $v(\theta_1, \theta_3)$ [which will become $v(\theta_1 - \theta_3)$ for a circular Fermi surface]. We say "nearly" in both cases because deviations of order Λ are kinematically allowed. These amplitude u will always be possible since if the two incoming lines lie in the bandwidth, (nearly) forward scattering will ensure the two final lines satisfy momentum conservation and lie in the bandwidth. The same logic works for v as long as we pick one incoming and one outgoing line in the bandwidth, for their opposites in the Cooper pair will satisfy momentum conservation and lie in the bandwidth as long as $E(-\mathbf{k}) = E(\mathbf{k})$.

By power counting the interactions u and v that survive the restriction to the narrow bandwidth near the Fermi surface scale as

$$v' = s^{-\varepsilon}v \qquad (4)$$

and similarly for u. Thus these are irrelevant for all codimension $d_c > 1$, and the low energy physics is that of the free fermion theory.

Let us now study a phase transition that occurs at finite interaction strength that gaps out all the fermions. We consider a superconducting transition driven by increasing v. We consider a simple model where v is constant except for the antisymmetry demanded by the Pauli principle:

$$S_4 = \frac{v}{4} \int \prod_{i=1}^{3} \int_{-\infty}^{\infty} \frac{d\omega_i}{2\pi} \int_{|\mathbf{k}|<\Lambda} d^{1+\varepsilon}\mathbf{k}_i$$
$$\times \int d\theta_i \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\bar{\psi}_\alpha(4)\bar{\psi}_\beta(3)\psi_\gamma(2)\psi_\delta(1), \qquad (5)$$

where labels like (2) are shorthand for angles, momenta and frequency labeled 2, and there is no integral over (4) which is required to equal $(1 + 2 - 3)$ by conservation laws. If v is large, the fermions will pair and a superconducting state will result. We now study the transition to this state to leading order in an ε expansion for small ε.

If we eliminate the modes between Λ and Λ/s we get an all too familiar loop correction

$$v' = s^{-\varepsilon}v(1 + v \ln s), \qquad (6)$$

where the loop may be done in $D = 2$ (which is $\varepsilon = 0$) since we expect the v in front of it to be of order ε at the fixed point. Thus

$$\beta(v) = \frac{dv}{d\ln s} = -\varepsilon v + v^2, \qquad (7)$$

with a fixed point

$$v^* = \varepsilon. \qquad (8)$$

We can now read off the correlation length exponent:

$$\nu = 1/\beta'(v^*) = 1/\varepsilon. \qquad (9)$$

We need to find the exponent that corresponds to the "magnetic field" that couples to the order parameter $\Delta(\omega, \mathbf{q}) = \langle(\psi\psi)\rangle_{\omega,\mathbf{q}}$. So let us consider a source term

$$S_h = \int_{|\mathbf{k}|<\Lambda} d^{1+\varepsilon}\mathbf{k} \int d\theta \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} h(\omega, \mathbf{q})\psi(\omega', \mathbf{k}, \theta)$$
$$\times \psi(-\omega' + w, -\mathbf{k} + \mathbf{q}, -\theta), \qquad (10)$$

where $-\theta$ is the angle on the Fermi surface where the momentum is opposite to that at the point θ.

In this expression, one field ψ lies inside the bandwidth and the other has the opposite frequency and momentum plus a tiny $\omega, \mathbf{q}$. It is readily verified that at tree level

$$h' = hs, \qquad (11)$$

independent of ε. At one loop, the source h couples to $\psi\psi$ via a particle-particle bubble (Fig. 1) whose momenta lie in the shell being eliminated. Doing the calculation at $\varepsilon = 0$
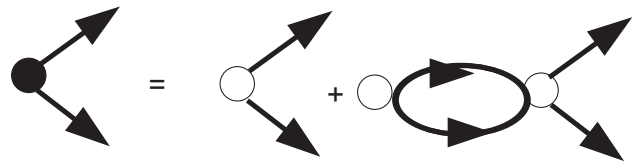

FIG. 1. Renormalization of the field that couples to the BCS order parameter at one loop. Open (closed) circle is the bare (renormalized) vertex.

due to the v^* that multiplies the bubble, we find

$$h' = hs(1 + v^* \ln s) \simeq hs^{1+\varepsilon}. \quad (12)$$

All we need now is the scaling law for E_{sing}, the singular part of the ground state energy per degree of freedom. Recall the argument. If we eliminate some high energy variables in a path integral, it will make an additive nonsingular contribution the free energy, to be carried on to the next stage of mode elimination. Since the singular part is the same, the singular part of the energy per degree of freedom will scale as

$$E_{\text{sing}}(\delta v, h) = s^{-D} E_{\text{sing}}(\delta v s^\varepsilon, h' s^{1+\varepsilon}), \quad (13)$$

where $\delta v = v - v^*$ and D is the number of dimensions along which we have thinned out degrees of freedom and rescaled frequency or momentum. The answer in our case is clearly $D = 2 + \varepsilon$. Henceforth the subscript in E_{sing} will be dropped. Thus we write

$$E(\delta v, h) = s^{-2-\varepsilon} E(\delta v s^\varepsilon, h s^{1+\varepsilon}). \quad (14)$$

By taking appropriate derivatives with respect to h and δv and then setting $s = (\delta v)^{-1/\varepsilon} = (\delta v)^{-\nu}$, $h = 0$, we find

$$\alpha = 1 - \frac{2}{\varepsilon} \qquad \gamma = 1 \qquad \eta = 2 - \varepsilon \qquad \beta = \nu = \frac{1}{\varepsilon}. \quad (15)$$

By α we mean the exponent for the second derivative of E with respect to the control parameter δv used to tune the transition. Note that $\alpha < 0$. The exponents β and γ come from the first and second derivatives with respect to h. One can find η, the anomalous exponent for the order parameter correlation, by directly computing it to one loop at $v^* = \varepsilon$ or by using the relation

$$\gamma = (2 - \eta)\nu. \quad (16)$$

We emphasize that the fermionic RG gives a canonical description of the critical point. In particular, conventional hyperscaling equations are satisfied with space-time dimensionality $d_c + 1$.

Let us ask what we would find had we chosen the Hertz-Millis route of integrating out the fermions to get an action for the pairing field Δ. We would find

$$S_\Delta = \int \Delta^*(\omega, \mathbf{q})[(q^2 + \omega^2)^{\varepsilon/2} + r]\Delta(\omega, \mathbf{q}) \frac{d\omega}{2\pi} d^{2+\varepsilon}\mathbf{q}$$
$$+ u\Delta^*\Delta\Delta^*\Delta \text{ terms} + \cdots. \quad (17)$$

Note that $\mathbf{q}$ lives in $2 + \varepsilon$ dimensions and we have a $D = 3 + \varepsilon$ theory. It has to reproduce a fixed point that we saw is of effective dimensionality $D = 2 + \varepsilon$. Let us see how far we can go with the quadratic part of the action, which is just the Cooper pair susceptibility.

First, the tree level scaling for the critical $r = 0$ theory is

$$\Delta = s^{(3+2\varepsilon)/2}\Delta', \quad \text{which means } h' = s^{(3+2\varepsilon)/2}h, \quad (18)$$

from which it follows that

$$r' = rs^\varepsilon, \quad \text{which means } \nu = \frac{1}{\varepsilon}, \quad (19)$$

in agreement with the old answer, even though Δ scales differently now.

From Eq. (18) it follows that the quartic coupling u in Eq. (17) is highly irrelevant if it is nonsingular in its ω and $\mathbf{q}$ dependence. But it is singular due to the fermion loop that generates it, but still irrelevant. Let us now calculate β paying attention to this dangerously irrelevant term.

If we evaluate the fermionic loop contribution to u, with all external $\mathbf{q}$ and ω equal to zero, we will find [from a box diagram of two pairs of fermions of equal and opposite $(\omega, \mathbf{k})$]

$$u \simeq \int \frac{d\omega d^{1+\varepsilon}\mathbf{k}}{(\omega^2 + k^2)^2}. \quad (20)$$

The infrared divergence of this diagram tells us that

$$u(\omega, \mathbf{q}) \simeq \frac{u_0}{(\sqrt{\omega^2 + q^2})^{2-\varepsilon}}. \quad (21)$$

From this and Eq. (17) we may deduce that

$$u_0(s) = s^{-1}u_0. \quad (22)$$

As in the case of ϕ^4 theory above four dimensions the u will be dangerously irrelevant and it will be necessary to keep it to obtain correct results for some exponents.

Turn now to the energy density which scales as follows:

$$E(r, h, u_0) = s^{-(3+\varepsilon)}E(rs^\varepsilon, hs^{(3/2+\varepsilon)}, u_0 s^{-1}). \quad (23)$$

The exponents ν, γ, and η can be deduced by working in the nonsuperconducting state (i.e., $r > 0$). Then we may safely set $u = 0$, and the results are identical to that obtained above by the fermionic RG.

The exponents β and α require retaining the irrelevant interactions. From the first derivative of E with respect to h we obtain the order parameter $m = \langle \Delta \rangle$:

$$m(r, 0, u_0) = s^{-(3+\varepsilon)}s^{((3/2)+\varepsilon)}m(rs^\varepsilon, 0, u_0 s^{-1}) \quad (24)$$

and the mean-field result far from the transition when rs^ε is of order unity,

$$m(rs^\varepsilon, u_0 s^{-1}) \simeq \sqrt{\frac{rs^\varepsilon}{u_0 s^{-1}}}. \quad (25)$$

Setting $rs^\varepsilon = 1$ we find $\beta = 1/\varepsilon$, again in agreement with the fermionic RG. Finally α (exponent for the second derivative with respect to the control parameter) may be obtained within mean-field theory by examining the ground state energy in the presence of a uniform condensate Δ which assumes the singular form

$$E = -r|\Delta|^2 + u|\Delta|^{2+\varepsilon} \quad (26)$$

in terms of r and u, which are viewed as fixed (and not

running) parameters. At the minimum $\Delta \simeq (r/u)^{1/\varepsilon}$ (so that $\beta = 1/\varepsilon$), while the second r derivative of the minimum energy gives $\alpha = 1 - \frac{2}{\varepsilon}$, in agreement with the fermionic RG. Note that the Gaussian action for Δ in the disordered phase describes the fluctuation correction which is subdominant to the mean-field answer.

The agreement of the fermionic and bosonic versions is remarkable and seems unlikely given that they describe theories in different number of effective dimensions ($2 + \varepsilon$ for fermions and $3 + \varepsilon$ for bosons). Despite this and the fact that Δ scales differently in the two versions, all exponents are the same order ε. It is always possible that something differs at higher orders in the ε expansion.

We believe the one-loop order ε results are exact. As stressed in Ref. [2] a narrow bandwidth problem has a small parameter Λ/K_F that plays the role of $1/N$, and in the large N limit the one-loop beta function is exact. Equivalently, as explained in Ref. [2], the RPA sum over repeated bubbles becomes exact because there is no phase space for any other diagram in the limit $\Lambda/K_F \to 0$.

Despite this agreement we emphasize that the fermionic approach is much more natural as it properly gives a scale invariant fixed point with a correct effective dimensionality of $d_c + 1$. In particular, all hyperscaling equalities are satisfied with this effective dimension. The Hertz-Millis theory does not, at first sight, look like a theory that will satisfy any such scaling. However, after accounting for the complications of the dangerously irrelevant interactions this hidden effective dimensionality is correctly captured. The Hertz-Millis description had to be more complicated in order for such a $(d + 1)$-dimensional theory to scale as though it lived in a different lower dimension $d_c + 1$. There also seem to be cases where the Hertz-Mills approach simply fails, at least in the simple form originally conceived [6,7].

The fate of the fermions themselves right at the critical point may be readily studied by considering their self-energy at the fixed point. We find that the fermionic quasiparticle residue Z stays nonzero all the way up to and including the critical point. The disappearance of the generalized Fermi surface on entering the superconducting state thus happens without ever destroying the fermionic quasiparticle. This is a different route to killing a Fermi surface from the Mott-like transitions considered in Ref. [9] where Z vanishes at the critical point [10]. Note that the Fermi surface stays sharp at the present critical point as expected on general grounds [9].

One can readily compute universal physics associated with the transition at finite temperatures to study the quantum critical region [11].

To conclude, we studied fermion systems with generalized "Fermi surfaces" with codimension d_c in d space dimension in the presence of interactions. Not surprisingly, weak short-range interactions were irrelevant for all $d_c > 1$. We studied a nontrivial superconducting transition out of such a phase through a controlled codimension expansion using the fermionic RG technique. The resulting theory scales with effective dimensionality $d_c + 1$. We also demonstrated that this result can be recovered within the Hertz-Millis approach despite the different apparent dimensionality. One can also hope to find the transition described here in real systems, for instance, in a three-dimensional superconductor with nodal lines as a "pairing" transition of the "Dirac-like" low energy quasiparticles.

More generally, studying phase transitions of Fermi surfaces with general codimension d_c in d space dimensions is a potentially useful theoretical device. In the future it might be interesting to understand the Stoner transition or problems with gauge fields by generalizing to $d_c > 1$. Perhaps such generalizations will lead to controlled expansion methods suitable for such problems.

[1] The RG for fermions was briefly addressed in G. Benfatto and G. Gallavotti, Phys. Rev. B **42**, 9967 (1990); J. Feldman and E. Trubowitz, Helv. Phys. Acta **63**, 157 (1990); R. Shankar, Physica (Amsterdam) **177A**, 530 (1991); J. Polchinski, arXiv:hep-th/9210046v2.

[2] The RG for fermions was addressed in great detail in R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).

[3] J. A. Hertz, Phys. Rev. B **14**, 1165 (1976).

[4] A. J. Millis, Phys. Rev. B **48**, 7183 (1993).

[5] T. Moriya, *Spin Fluctuations in Itinerant Electron Magnetism* (Springer-Verlag, Berlin, 1985).

[6] A. Abanov and A. Chubukov, Phys. Rev. Lett. **93**, 255702 (2004).

[7] D. Belitz, T. R. Kirkpatrick, and T. Vojta, Rev. Mod. Phys. **77**, 579 (2005).

[8] Liang Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett.,**98**, 106803 (2007).

[9] T. Senthil, Phys. Rev. B**78**, 035103 (2008).

[10] W. F. Brinkman and T. M. Rice, Phys. Rev. B **2**, 4302 (1970).

[11] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).