Magnetohydrodynamic Stability of a Toroidal Plasma's Separatrix

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Large tokamaks capable of fusion power production such as ITER, should avoid large edge localized modes (ELMs), thought to be triggered by an ideal magnetohydrodynamic instability due to current at the plasma's separatrix boundary. Unlike analytical work in a cylindrical approximation, numerical work finds the modes are stable. The plasma's separatrix might stabilize modes, but makes analytical and numerical work difficult. We generalize a cylindrical model to toroidal separatrix geometry, finding one parameter Δ' determines stability. The conformal transformation method is generalized to allow nonzero derivatives of a function on a boundary, and calculation of the equilibrium vacuum field allows Δ' to be found analytically. As a boundary more closely approximates a separatrix, we find the energy principle indicates instability, but the growth rate asymptotes to zero.

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Introduction.—Tokamak experiments allow controlled fusion of deuterium and tritium nuclei in a toroidal axisymmetric magnetically-confined plasma, and offer a potential technology for fusion power production. However, plasma instabilities at the plasma's edge (ELMs), could significantly erode plasma facing components in a large device such as ITER [1]. Our understanding of ELMs is based on an ideal magnetohydrodynamic (MHD) instability, but numerical studies are sensitive to approximations near the separatrix separating plasma from vacuum [2], and there has been little analytic progress. At fixed toroidal angle, the separatrix of modern tokamak plasmas has a cross sectional shape with at least one point at which it forms a sharp, typically 90° angle where the poloidal (but not toroidal), magnetic field component becomes zero. This X point makes the geometry difficult to study analytically, and causes numerical problems.

Careful numerical work [3] has nonetheless suggested that a separatrix at the plasma-vacuum boundary can stabilize the instability that is thought to trigger ELMs. This is surprising because the strong current gradient at the plasma edge that provides the drive for the instability is unaffected by a separatrix. Furthermore, analytical work [4] in a cylindrical approximation found the mode always unstable, regardless of cross-sectional shape, though it did not consider how the growth rate might be affected as a separatrix shape is approximated.

Here we study the stability of edge-current driven modes in a plasma cross section with a separatrix. We tackle the underlying physical and mathematical difficulties, namely, we: (i) establish a simple toroidal MHD model for edge-current driven modes that does not couple to pressure-driven modes, (ii) analytically calculate equilibrium vacuum fields in a separatrix geometry, (iii) analytically calculate (for high toroidal mode number), the parameters δW and Δ' (defined later), needed to determine the mode's (in) stability. This Letter is intended to serve as a summary and introduction to a full and detailed treatment to be presented elsewhere.

The Peeling mode.—The stability of an ideal magnetohydrodynamic (MHD) plasma equilibrium is determined by the competition between the stabilizing influence of: the magnetic field, plasma compressibility, and the energy associated with the perturbation to the magnetic field of any surrounding vacuum, and the destabilizing influence of: the pressure gradient, current gradients, and current in the plasma. This may be seen from the energy principle's [5] δW, from which stability is usually determined by the sign of $\delta W = \delta W_F + \delta W_S + \delta W_V$, where δW_V is the perturbed energy of the surrounding vacuum, δW_F contains a stabilizing contribution from the magnetic field and the plasma compressibility, and a destabilizing contribution from the pressure gradient and the plasma's current, δW_S is a surface integral that was obtained by integrating by parts with respect to radius, and implicitly contains a destabilizing drive from the current gradient. At a boundary between a current-carrying plasma and a vacuum, the current gradient can be especially strong, because the current reduces to zero in the vacuum. This suggests the possibility for a radially localized mode at the plasma's edge, whose stability is solely determined by the competition between the destabilizing drive from a strong current gradient at the plasma-vacuum boundary, and the stabilizing influence of the magnetic field at the edge and in the surrounding vacuum. This mode is often called a "Peeling" or "external kink" mode, respectively referring to its radial localization and the need for a current to drive it unstable (like the internal "kink" mode). This physical description of a Peeling mode (PM) suggests that its stability is determined by $\delta W = \delta W_S + \delta W_V$, where δW_S contains the strongest edge-localized contributions to δW, so that the radial localization of the mode then allows δW_F to be neglected. By considering a generalization of a simple cylindrical geometry PM model to toroidal geometry in a high toroidal mode number limit n, this is explicitly shown to be the case, as outlined below.

We linearize the ideal MHD equations about a cylindrical plasma equilibrium, with the approximation of a plasma surface marking a sharp transition from plasma to vacuum, and integrate the resulting equations radially from just inside the plasma surface to the vacuum just outside. A plasma perturbation produces a perturbed magnetic field with a discontinuous radial derivative across the plasma surface. This discontinuity, and the strong current gradient at the plasma's surface both give non-negligible contributions to the equations. For cylindrical polar coordinates (r, θ, ϕ), with m and n poloidal and toroidal mode numbers, respectively, and a plasma displacement $\vec{\xi} \sim \vec{\xi}_0 e^{im\theta - in\phi}$, marginal stability requires [6]

$$0 = \Delta\Delta' + J_{\parallel} \tag{1}$$

with $J_{\parallel} = \frac{r}{B_p}\frac{\vec{B}\cdot\vec{J}}{B}$, $\Delta = \frac{m-nq}{m}$, $\Delta' = \lim_{\epsilon\to 0}[\frac{r}{b_r}\frac{db_r}{dr}]^{+\epsilon}_{-\epsilon}$ with $-\epsilon$ in the plasma, $+\epsilon$ in the vacuum, q is the "safety factor" [5] (the number of times a field line goes toroidally around the plasma for one poloidal turn), and b_r is the radial component of the perturbed magnetic field. (m and n may be chosen to ensure the equation is satisfied to arbitrary accuracy.)

In the high-n limit, we have generalized this calculation to toroidal geometry, considering an axisymmetric toroidal system with orthogonal coordinate system [5] (ψ, χ, ϕ) Jacobian J_χ, and equilibrium magnetic field [5] $\vec{B} = I(\psi)\vec{\nabla}\phi + \vec{\nabla}\phi \times \vec{\nabla}\psi$, with ψ the poloidal magnetic flux. To generalize the trial function of $\vec{\xi} \sim \vec{\xi}_0 e^{im\theta - in\phi}$ to toroidal geometry, we replace the cylindrical angle θ with $\theta = \frac{1}{q}\int^{\chi} \nu d\chi$, where $\nu = IJ_\chi/R^2$ is the local field-line pitch, and R the major radius. This is the same trial function used by Laval *et al.* [4]. Generalizing the cylindrical calculation to toroidal geometry, we find that at high-n marginal stability is identical to solving $0 = \delta W_S + \delta W_V$, with δW_S as given in Ref. [7]. Therefore we define the PM by $\delta W_F \ll \delta W_S + \delta W_V$, and the minimization of $\delta W = \delta W_S + \delta W_V$. The trial function gives

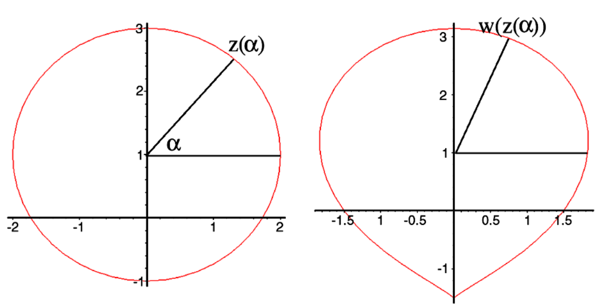

FIG. 1 (color online). The Karman-Trefftz function [9] maps from a circular boundary to a separatrix. The angle α parameterizes the position $w(z(\alpha))$ on the separatrix.

$$\delta W = -2\pi^2 \frac{|\xi_m|^2}{R_0}\Delta(\Delta\Delta' + \hat{J}), \tag{2}$$

the same form as the cylindrical case [6], but with

$$\Delta' = \lim_{\epsilon\to 0}\left[\frac{1}{2\pi}\oint dl R_0 B_p \frac{I^2}{R^2 B_p^2}\frac{\frac{\partial}{\partial\psi}(\vec{\nabla}\psi\cdot\vec{B}_1)}{\vec{\nabla}\psi\cdot\vec{B}_1}\right]^{+\epsilon}_{-\epsilon}, \tag{3}$$

$$\hat{J} = \frac{1}{2\pi}\oint dl \frac{IR_0}{R^2 B_p}\frac{\vec{J}\cdot\vec{B}}{B^2} \tag{4}$$

and $\Delta = \frac{m-nq}{nq}$, $\xi_\psi = \vec{\nabla}\psi\cdot\vec{\xi} = \xi_m(\psi)e^{im\theta - in\phi}$ the trial function, R_0 a measure of the major radius, and dl an infinitesimal element of arc length in the plasma surface at fixed toroidal angle. This extends previous work in allowing arbitrary currents at the plasma surface and arbitrary toroidal geometries. It also gives a simple stability criterion for edge-current driven PMs, without coupling to pressure-driven modes, for example. The decoupling from pressure-driven modes is possible because PMs neglect δW_F. The integrals of Eqs. (3) and (4) are dominated by the divergence in $1/B_p$ at the X point, hence toroidal effects only appear through the X point's position determining R when the integrand is large. Minimizing δW with respect to Δ (or equivalently n), gives

$$\delta W = \frac{\pi^2}{2}\frac{|\xi_m|^2}{R_0}\left(\frac{\hat{J}^2}{\Delta'}\right) \tag{5}$$

so that stability will be determined by the value of Δ'.

Evaluating Δ'.—In the PM model, δW is determined by quantities at the plasma's surface. At equilibrium we assume that the magnetic fields in the vacuum $\vec{B}^V$, and the plasma $\vec{B}$, are equal at the plasma surface. The angle $\theta = \frac{1}{q}\int^{\chi}\nu d\chi$ may be written as an integral along the plasma surface at fixed toroidal angle with $\theta = \frac{1}{q}\int^{l}\frac{I}{R^2}\frac{dl}{B_p}$. Therefore at the plasma surface θ is determined by $\vec{B}^V$. More generally, we will find that for the large aspect ratio limit Δ' and δW are determined solely by the equilibrium and perturbed vacuum fields. This is important because vacuum fields satisfy Laplace's equation, and in a large aspect ratio limit the system is approximately two dimensional; properties that allow a conformal transformation [8] to be used.

Conformal transformations are mappings by an analytic function, and ensure that a function satisfying Laplace's equation will continue to do so in the transformed coordinates. Textbook methods require either the function or its normal derivative to be zero on the boundary. This greatly simplifies the calculation, allowing an analytic solution to be obtained; but we find that this is not essential. The magnetic field in the vacuum has $\vec{\nabla}\times\vec{B}^V = 0$ and $\vec{\nabla}\cdot\vec{B}^V = 0$, and hence may be written as the gradient of a scalar with $\vec{B}^V = \vec{\nabla}V$ and $\nabla^2 V = 0$. Under an analytic map with $z \rightarrow w(z)$, we find the normal component of the field on the boundary transforms according to $n_z \cdot B_z = |\frac{dw}{dz}| n_w \cdot B_w$ where the unit normals $n_{z,w}$ and fields $B_{z,w}$ are complex numbers interpreted as vectors in the usual way [8], the subscripts indicate their evaluation in the circular cross-section system (z) or the separatrix boundary (w). The dot product refers to the sum of the products of the real and imaginary components, respectively. Now, instead of directly obtaining an analytical solution (the case if the function or its normal derivative are zero on the boundary), a two-dimensional problem is reduced to one dimension. We illustrate the method by outlining the calculation of δW_V.

It is well known [8] that the field B_z and the volume element $\vec{dr}_z$ transform as $B_w = \overline{\frac{dz}{dw}} B_z$ and $\vec{dr}_w = |\frac{dw}{dz}|^2 \vec{dr}_z$. Therefore, $\delta W_V = \int |B_w|^2 \vec{dr}_w = \int |B_z|^2 \vec{dr}_z$, giving the vacuum energy in terms of the vacuum energy for the circular cross-section system. Solutions for the vacuum field in the circular cross-section system give δW_V as a sum of Fourier coefficients with $\delta W_V = 2\pi^2 R_0 \sum_{p\neq 0} |p||a_p|^2$. The coefficients a_p are determined from the boundary conditions. The boundary conditions for the circular cross-section system are found from $n_z \cdot B_z = |\frac{dw}{dz}| n_w \cdot B_w$ and the plasma-vacuum boundary conditions of $\vec{\nabla}\psi \cdot \vec{B}_1^V = \vec{\nabla}\psi \cdot \vec{B}_1$. In our notation $RB_w n_w \cdot B_w = \vec{\nabla}\psi \cdot \vec{B}_1^V$, and we use $\vec{\nabla}\psi \cdot \vec{B}_1 = \vec{B} \cdot \vec{\nabla}\xi_\psi$, and the trial function $\xi_\psi = \xi_m(\psi) e^{im\theta(\alpha) - in\phi}$ whose poloidal dependence is parameterized by the angle α in the circular boundary system. Projecting out the Fourier coefficients gives

$$a_p = \frac{ip}{|p|} \Delta \frac{\xi_m}{R_0} \frac{1}{2\pi} \oint e^{im\theta(\alpha) - ip\alpha} d\alpha. \quad (6)$$

To obtain $\theta(\alpha)$ we use a conformal map to calculate an equilibrium vacuum field for a separatrix equilibrium. By combining a vertical and a circulating field, a vacuum field with a circular boundary on which the field is zero at a single point may be produced. Using the Karman-Trefftz [9] function to transform the field to one with a separatrix boundary (Fig. 1), gives a vacuum field for which the boundary condition of $\vec{n} \cdot \vec{B} = 0$ is satisfied, with $\vec{B}_p = 0$ at the X point (Fig. 2).

The poloidal field at the separatrix is parameterised by α in the circular cross-section system, leading to analytic expressions for $\theta(\alpha)$ and q. Near the separatrix $\theta(\alpha) \simeq \frac{c_t}{q} \int_{-\pi}^{\alpha} \frac{d\alpha}{\sqrt{\alpha^2/2 + \epsilon^2/2a^2}}$ and $q \simeq \frac{c_t\sqrt{2}}{\pi} \ln(a/\epsilon)$, c_t is a constant and $\epsilon/a \rightarrow 0$ as we more closely approximate a separatrix. We now have all the analytical expressions needed to calculate the vacuum energy δW_V. We may write the sum as $\sum_{p=-\infty}^{\infty} p|a_p|^2 + 2\sum_{p=-\infty}^{-1} |p||a_p|^2$ and exactly resum the first expression for which $\sum_{p=-\infty}^{\infty} p|a_p|^2 = \Delta^2 \frac{|\xi_m|^2}{R_0^2} m$ noting that we expect $|a_p|^2$ to be peaked near $p \sim m \sim nq \gg 1$ and hence $\sum_{p=-\infty}^{-1} |p||a_p|^2 \ll \sum_{p=-\infty}^{\infty} p|a_p|^2$. The calculation and its approximations have been confirmed by a saddle point approximation. Repeating the calculation with a simple piecewise linear function for $\theta(\alpha)$, that tends to a step function as we approach the separatrix, gives the same answer. This suggests that for large mode numbers the result is generic and independent of the detailed form of $\theta(\alpha)$.

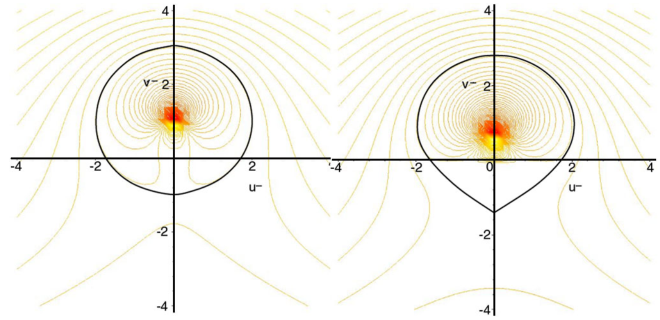

FIG. 2 (color online). A vertical and rotating field can be combined to form a field that is zero at a point on a circular boundary [8,9] (left figure). A Karman-Trefftz transformation maps the boundary to a separatrix, with zero field at the X point (right figure).

We obtain Δ' similarly, but also use $n_z \cdot \nabla_z = |\frac{dw}{dz}| n_w . \nabla_w$, an expression tested by calculating δW_V from its surface integral representation. The details of the calculation will be described in detail elsewhere, but the result is that $\Delta' = -2m$, the same as for an equivalent perturbation in a circular cross-section system.

Peeling mode stability.—For $\Delta' = -2m$, Eq. (5) for δW is clearly negative, consistent with previous analytical work [4]. This is usually taken to indicate instability. However the growth rate γ determines how unstable the mode is, from [5]

$$\gamma^2 = -\delta W / \int \rho_0 |\xi|^2 \vec{dr}. \quad (7)$$

The MHD modes with minimum δW have [5], $\vec{\nabla} \cdot \vec{\xi} = 0$, requiring $\xi_\perp = \frac{i}{n} \vec{\nabla}\psi \cdot \vec{\nabla}\xi_\psi$, where $\vec{\xi} = \xi_B \vec{B}/B^2 + \xi_\perp \vec{B} \times \vec{\nabla}\psi / R^2 B_p^2 B^2 + \xi_\psi \vec{\nabla}\psi / R^2 B_p^2$. However, to keep the field-line bending of order 1, a mode must oscillate $m \sim nq$ times, requiring $\xi_\perp \propto q' \xi_\psi$, as may also be seen by differentiating the trial function. If we approximate $|\xi_m|$ as a power law and assume q has a logarithmic dependence on ψ near a separatrix (found here and elsewhere), then writing $\vec{dr} = \frac{dl}{B_p} d\psi d\phi$ gives

$$\int \rho_0 |\xi|^2 \vec{dr} \propto \int |\xi_m|^2 q'^2 d\psi \sim |\xi_m|^2 q' \quad (8)$$

Thus the rapid mode oscillations near an X point and

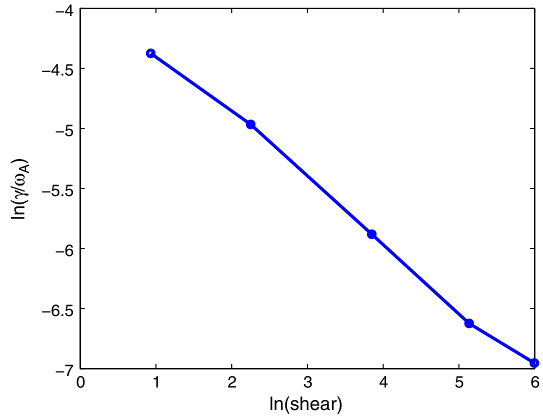

FIG. 3 (color online). ELITE [10,11] has many Fourier modes, but a suitable equilibrium allows study of PM stability. At high shear, Eq. (9) becomes accurate, but ELITE's calculations become challenging. Nonetheless, $\ln(\gamma)$ is approximately proportional to $\ln(q'/q)$, with slope -0.52 close to the predicted -0.5.

$\vec{\nabla}\cdot\vec{\xi} \lesssim 1$, leads to a kinetic energy that diverges at a rate proportional to q'.

To calculate the growth rate we take $\vec{\nabla}\phi\cdot\vec{J} \neq 0$ and approximate $\hat{J}$ by $\hat{J} \simeq R_0^3 \frac{\vec{\nabla}\phi\cdot\vec{J}}{2\pi I} \oint \frac{I}{R^2}\frac{dl}{B_p} = qR_0^3\vec{\nabla}\phi\cdot\vec{J}/I$. Then with $\Delta' = -2m \simeq -2nq$ we find $\hat{J}^2/\Delta' \sim q/n$. The divergence in $\hat{J}$ is due to the normalization of $\xi_\psi \sim RB_p|\vec{\xi}|$, that is used in many calculations and codes including ELITE [10], the effects of which will cancel when the growth rate is calculated. We use $\hat{J}^2/\Delta' \sim q/n$ in Eq. (5) and substitute this and Eq. (8) into Eq. (7) for the growth rate, then with $\gamma_A^2 \equiv B^2/(\rho_0 R \oint dl)$, we find

$$\ln(\gamma/\gamma_A) = -\tfrac{1}{2}\ln(s) \quad (9)$$

for a boundary that approximates a separatrix with a magnetic shear $s \propto q'/q$ that tends to infinity. This has subsequently been compared with results from ELITE [11], and excellent agreement is found (Fig. 3).

Whereas δW is minimized by $\vec{\nabla}\cdot\vec{\xi} = 0$, minimization of δW does not necessarily maximize the growth rate. However, if we allow $\vec{\nabla}\cdot\vec{\xi} \neq 0$ so as to keep the kinetic energy term of order 1, then we instead find $\vec{\nabla}\cdot\vec{\xi} \sim q'$, and the plasma compressibility is strongly stabilizing. Therefore the strong deformation of flux tubes near an X point either leads to a diverging kinetic energy or strong stabilization through $\vec{\nabla}\cdot\vec{\xi} \rightarrow \infty$. This could be avoided by a mode that becomes zero near the X point, but that will increase the stabilization from field-line bending.

Conclusions.—For high-n we have generalized an ideal MHD model for PMs to toroidal geometry. It is not complicated by coupling to pressure-driven modes, for example, and is in quantitative agreement with ELITE. A quantitative comparison was possible by an analytical calculation of δW using conformal transformations in a more general way than is usual. Our model allows us to explicitly see that whereas the drive for instability remains (as physically expected and reflected in us finding $\delta W < 0$ as for previous analytical work), the kinetic energy diverges and causes the growth rate $\gamma/\gamma_A \rightarrow 0$ (consistent with numerical calculations). The physical origin of this is the strong deformation of flux tubes near an X point, that either strongly stabilize due to the plasma's compressibility (that is eliminated in ideal MHD studies that minimise δW), or to a diverging kinetic energy. Therefore we believe the stabilizing effect of an X point on the high-n ideal MHD model is now understood. Future work will consider how nonideal effects such as resistivity (that can reduce fieldline bending, for example), will modify this picture.

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