Monogamy of Bell's Inequality Violations in Nonsignaling Theories

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We derive monogamy relations (tradeoffs) between strengths of violations of Bell's inequalities from the nonsignaling condition. Our result applies to general Bell inequalities with an arbitrary large number of partners, outcomes, and measurement settings. The method is simple, efficient, and does not require linear programing. The results are used to derive optimal fidelity for asymmetric cloning in nonsignaling theories.

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The nonsignaling principle—the impossibility of sending information faster than the speed of light—is deeply rooted in our existing understanding of the physical world. It not only allows us to consider current physical theories within a general framework of the nonsignaling principle, but also to significantly restrict the structure of possible future theories. This principle implies that the correlations between distant partners cannot be used to send information, as is the case for quantum correlations. Mathematically, a correlation is defined as a joint probability distribution $P(a, b|x, y)$, where a and b are outcomes of two separated parties, say Alice and Bob, given x and y as their free choices of measurement settings, respectively. The nonsignaling condition implies that the marginals are independent of the partner's choice: $P(a|x, y) = \sum_b P(a, b|x, y) = P(a|x)$.

Quantum theory predicts correlations between spacelike separated events, which are nonsignaling but cannot be explained within local realism, i.e., within the framework in which all outcomes have preexisting values for any possible measurement before the measurements are made ("realism") and where these values are independent from any action at spacelike separated regions ("locality") [1]. This is signified by the violation of Bell's inequalities. Since the work of Popescu and Rohrlich [2], it is known that there are correlations violating Bell's inequality stronger than the quantum mechanical correlations, but without contradicting the nonsignaling principle. This opened up a possibility to investigate quantum correlations outside of the Hilbert space formalism as well as correlations in general probabilistic theories subject to the nonsignaling constraint [3–7].

The general framework for considering nonsignaling correlations is also important from the information-theoretical point of view. For example, protocols for a secret key distribution were recently proposed and their security proved solely using the nonsignaling principle [8,9]. Furthermore, it was shown that every nonsignaling theory that predicts the violation of Bell's inequality implies the no-cloning theorem. The bound on the shrinking factor for the symmetric, phase-covariant cloning was derived from the nonsignaling condition [5,10].

In this Letter, we will investigate monogamy properties of correlations in nonsignaling theories. This property was first found for quantum entanglement. Consider, for example, three subsystems A, B, and C of a composite quantum system. The theorem of Coffman, Kundu, and Wootters describes the trade-off between the degree of entanglement between A and B and the degree of entanglement between A and C, as measured by concurrence [11–14]. A similar tradeoff exits between the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality for the pair A-B and the violation of the inequality for the pair A-C in *any* nonsignaling theory [5,15] (For the tradeoff derived within quantum theory, see Ref. [16,17]). The questions arise: Is the monogamy relation a generic feature of *every* Bell inequality? What are constraints on quantum correlations imposed by the nonsignaling condition? A general, but only qualitative result was found [5]: If A and B maximally violate some Bell inequality, then A and C are completely uncorrelated. Furthermore, a linear program was given for finding the nonsignaling bounds on the quantum value of a general Bell expression [15].

Here, we derive the monogamy relations for the violation of *general* Bell's inequalities in any nonsignaling theory. It applies for an *arbitrary* number of parties, measurement settings, and outcomes. The method is simple, efficient, and does not require linear programing. To illustrate its applicability, we derive the optimal fidelity for generally asymmetric cloning from the nonsignaling bounds. The latter generalizes the results of Ref. [5]

Consider a general linear, two-partite Bell inequality, for correlations of local outcomes observed at measurement stations of Alice (A) and Bob (B),

$$\mathcal{B}(A, B) \equiv \sum_{x,y} \sum_{a,b} \alpha(x, y, a, b) P(A_x = a, B_y = b) \leq R. \tag{1}$$

Here, x and y stand for the measurement settings chosen by Alice and Bob, respectively, and a and b for the outcomes of their measurements. R is the local realistic bound and $P(A_x = a, B_y = b) \equiv P(a, b|x, y)$ is the conditional probability (both notations will be used in the present work).

Throughout this Letter, we will assume that every Bell inequality is written in such a form that for all x, y, a, b, one has $\alpha(x, y, a, b) \geq 0$. This guarantees that $\mathcal{B}(A, B) \geq 0$ and $R \geq 0$. To see that every inequality can be brought in this form, note that each inequality which has some negative α's can be rewritten by substituting probabilities which are next to negative α's by unity minus the probability of the opposite events. The chosen form simplifies the formulas for Bell's inequalities as no absolute values need to be involved.

We now give the main result of our Letter. Consider $n + 1$ separated parties, a single Alice (A) and a set of n Bobs ($\mathrm{B}^{(1)}, \ldots, \mathrm{B}^{(n)}$). Furthermore, consider a linear bipartite Bell's inequality $\mathcal{B}(A, B^{(m)}) \leq R$ of type (1), for measurements of A and any single Bob $B^{(m)}$, $m \in \{1, \ldots, n\}$. The number of outcomes at the two stations is arbitrary, as well as the number of measurement settings at A. The number of settings at each $B^{(m)}$ is assumed to be n, which is also the total number of Bobs. The following monogamy relation must hold between the strengths of violations of bipartite Bell's inequalities for n pairs of observers, each pair consisting of Alice and single Bob:

$$\sum_{m=1}^{n} \mathcal{B}(A, B^{(m)}) \leq nR. \tag{2}$$

This holds in every nonsignaling theory, including these for which individual Bell's inequalities $\mathcal{B}(A, B^{(m)}) \leq R$ can be violated, as it is the case in quantum theory (An analogous result of Eq. (2) within quantum theory was found in Ref. [18]).

The proof consists in showing that a violation of the monogamy relation (2) would imply signaling. The left-hand side of Ineq. (2) can be written as $\sum_{m=1}^{n} \mathcal{B}(A, B^{(m)}) = \sum_{m=1}^{n} \mathcal{B}_m$, where

$$\mathcal{B}_m \equiv \sum_{x,y} \sum_{a,b} \alpha(x, y, a, b) P(A_x = a, B_y^{(y+m-1 \bmod n)} = b) \tag{3}$$

involves a sum over all the settings of Alice and only one setting for each Bob (see Fig. 1). Here, $P(A_x = a, B_y^{(y+m-1 \bmod n)} = b)$ is the probability that Alice observes a and the $(y + m - 1 \bmod n)$-th Bob observes b, when she chooses setting x and he setting y. If Ineq. (2) is violated, then there exists at least one m for which

$$\mathcal{B}_m \leq R \tag{4}$$

is violated. We show that violation of Ineq. (4) implies signaling. We prove it for $m = 1$; for other m values, the proof is analogous. The Ineq. (4) for $m = 1$ reads

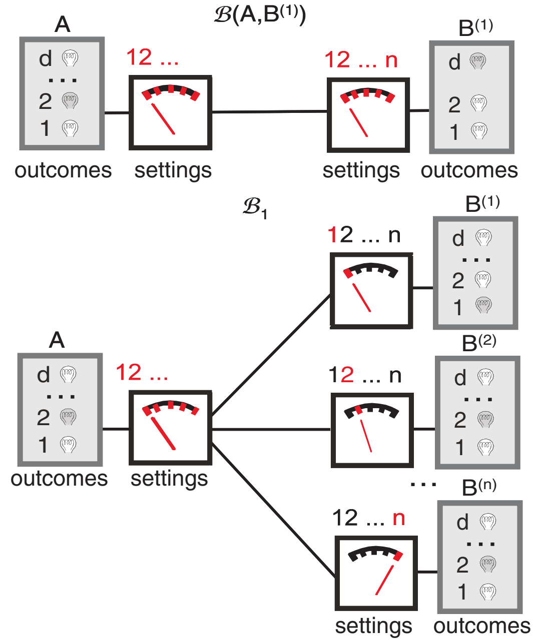

FIG. 1 (color online). Diagram of measurements involved in the Bell expression $\mathcal{B}(A, B^{(1)})$ (top) and $\mathcal{B}_1$ (bottom). The choices of the measurement settings are marked in red (online) and by positions of the pointers (print). In the setup for $\mathcal{B}(A, B^{(1)})$, both parties have a number of measurements to (freely [21]) choose from. In the setup for $\mathcal{B}_1$ only Alice has such a choice whereas each $B^{(y)}$, $y \in \{1, \ldots, n\}$ always performs the same measurement y.

$$\mathcal{B}_1 \equiv \sum_{x,y} \sum_{a,b} \alpha(x, y, a, b) P(A_x = a, B_y^{(y)} = b) \leq R, \tag{5}$$

and is again a Bell's inequality of type (1).

It is important to note that in the present setup, Bobs do not change their measurement settings during the Bell test and, furthermore, that they all can jointly perform their measurements. Thus, observer $B^{(1)}$ always performs measurement 1, and simultaneously $B^{(2)}$, performs measurement 2 and so on. Let us introduce the joint probability $P(A_x = a, B_1^{(1)} = b_1, \ldots, B_n^{(n)} = b_n)$ that Alice observes the outcome a when she chooses the setting x, and Bobs observe sequence of outcomes $b_1, \ldots, b_n$. We write $P(A_x = a, B_y^{(y)} = b) = \sum'_{b_1,\ldots,b_n} P(A_x = a, B_1^{(1)} = b_1, \ldots, B_y^{(y)} = b, \ldots, B_n^{(n)} = b_n)$, where $\sum'_{b_1,\ldots,b_n}$ denotes the sum over all indices $b_1, \ldots, b_n$ except b_y. The Ineq. (5) can now be brought into the form,

$$\mathcal{B}_1 = \sum_{x,a} \sum_{b_1,\ldots,b_n} \alpha'(x, a, b_1, \ldots, b_n) \times P(A_x = a, B_1^{(1)} = b_1, \ldots, B_n^{(n)} = b_n) \leq R, \tag{6}$$

where $\alpha'(x, a, b_1, \ldots, b_n) = \sum_y \alpha(x, y, a, b_y)$.

We introduce the short notation $\vec{b} \equiv (b_1, \ldots, b_n)$ for the set of all outcomes that are observed by Bobs and $P(a, \vec{b}|x) \equiv P(A_x = a, B_1^{(1)} = b_1, \ldots, B_y^{(y)} = b, \ldots, B_n^{(n)} = b_n)$ for the probability that Alice observes a and Bobs $\vec{b}$ conditional on her choice of setting x. Recall that these probabilities are not conditioned on the choice of the measurement settings of Bobs since in the setup considered ($\mathcal{B}_1$) all settings y are chosen simultaneously by different Bobs. We now can rewrite Ineq. (6) as

$$\mathcal{B}_1 = \sum_{x,a,\vec{b}} \alpha'(x, a, \vec{b}) P(a, \vec{b}|x) \leq R. \tag{7}$$

For every probability distribution, it is valid that

$$P(a, \vec{b}|x) = P(a|\vec{b}, x) P(\vec{b}|x). \tag{8}$$

The nonsignaling condition is the assumption that

$$P(\vec{b}|x) = P(\vec{b}), \tag{9}$$

which allows to write Eq. (8) as

$$P(a, \vec{b}|x) = P(a|\vec{b}, x) P(\vec{b}). \tag{10}$$

It is crucial to realize that a probability distribution that satisfies Eq. (10) is explainable within local realism. In a local realistic model, the source sends particles carrying information about the vector $\vec{b}$ with the probability $P(\vec{b})$ to Alice and all Bobs. The measurement apparatuses of Bobs output $\vec{b}$ while the apparatus of Alice takes the input x (the setting chosen freely by Alice) and outputs a with the probability $P(a|\vec{b}, x)$. This means that every value of the left-hand side of Ineq. (7), which is attainable by any nonsignaling theory, is also attainable by a local realistic one. And since R is the maximal attainable value of the left-hand side of Ineq. (7) this in turn means that a violation of Ineq. (7) would allow Alice to signal to Bobs.

The extension to multipartite Bell's inequalities is straightforward. Consider Bell's inequality

$$\mathcal{B}(P^{(1)}, \ldots, P^{(N)}) \leq R \tag{11}$$

where N parties $P^{(i)}$, $i \in \{1, \ldots, N\}$, can choose between an arbitrary number of measurement settings. We can always divide the parties into two sets and name these sets $\vec{A}$ and $\vec{B}$. We can now consider each of these two sets as one party in a corresponding two-partite Bell's inequality and rewrite Ineq. (11) as $\mathcal{B}(\vec{A}, \vec{B}) \leq R$. Each setting of $\vec{A}$ and $\vec{B}$ corresponds to one of all the possible combinations of settings of individual parties that form the set. Following the proof given above, we can conclude: For every N-partite Bell inequality $\mathcal{B}(P^{(1)}, \ldots, P^{(N)}) \leq R$ and any chosen division of the parties into two sets $\vec{A}$ and $\vec{B}$, the violation of

$$\sum_{m=1}^{n} \beta(\vec{A}, \vec{B}^{(m)}) \leq nR, \tag{12}$$

where n is the number of settings at each $\vec{B}^{(m)}$ (the number of the settings at $\vec{A}$ is arbitrary), implies signaling.

To illustrate the consequences of our result, we will consider an asymmetric, state dependent cloning machine (see Ref. [19] for a review on cloning) that takes a single system of arbitrary dimension and produces n copies ($1 \rightarrow n$ cloning machine). We will derive the optimal shrinking factor for the machine from our nonsignaling inequalities (2). Consider a composite system consisting of two subsystems belonging to A and B. The two subsystems can be measured locally giving rise to the probability distribution $P(A_x = a, B_y = b)$ (Fig. 2, top). Alternatively, the subsystem of B can be sent to the cloning machine which takes it as an input and outputs n "copies" that are further distributed to n observers $B^{(m)}$, $m \in \{1, \ldots, n\}$, and then measured locally in coincidence with the subsystem of A. The "cloned" probability distribution for local measurements on A and $B^{(m)}$ is denoted by $P(A_x = a, B_y^{(m)} = b)$. Given an initial probability distribu-

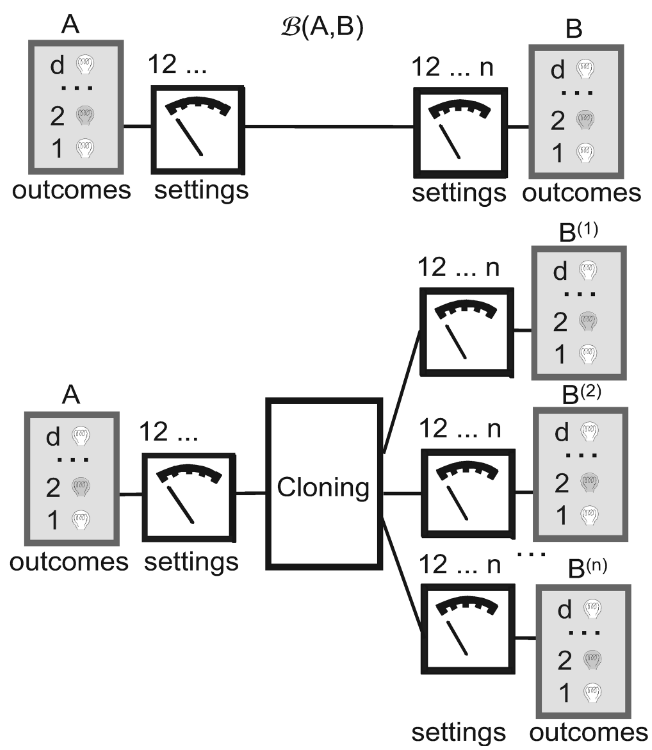

FIG. 2. Diagram of measurements involved in a direct Bell test between Alice and Bob (top) or the one between Alice and n Bobs after the cloning procedure (bottom). While Alice chooses between an arbitrary number of measurement settings, Bobs choose between n of them. The nonsignaling condition gives an upper bound of the shrinking factor for a general asymmetric cloning procedure employed by Bobs.

tion, which cloned probability distributions are in agreement with the nonsignaling condition?

We now compare the strengths of the violation of Bell's inequality on an arbitrarily dimensional composite system before and after the cloning procedure. Denote the Bell expression in the experiment without cloning by $\mathcal{B}(A, B)$. Alice is assumed to choose between an arbitrary number of measurement settings and Bob between n of them. Denote, furthermore, the Bell expressions in the experiment with cloning by $\mathcal{B}(A, B^{(m)})$, $m \in \{1, \ldots, n\}$. Every such expression involves the cloned probability distribution between a pair of observers A and $B^{(m)}$, where A chooses among an arbitrary number of measurement settings, and each $B^{(m)}$ between n of them. We define the mean value of the shrinking factor η_m for each of the copies to be

$$\eta_m = \frac{\mathcal{B}(A, B^{(m)})}{\mathcal{B}(A, B)}. \tag{13}$$

The nonsignaling inequality (2) implies

$$\sum_{m=1}^{n} \mathcal{B}(A, B^{(m)}) \leq nR, \tag{14}$$

which transforms into

$$\frac{1}{n} \sum_{m=1}^{n} \eta_m \leq \frac{R}{\mathcal{B}(A, B)}. \tag{15}$$

Therefore, the bound on the mean value of the shrinking factor for cloning is nontrivial, i.e., less than unity, only if the initial probability distribution *violates* Bell's inequality. This generalizes the results of Ref. [5] obtained for the symmetric cloning and $n = 2$.

The bound derived with the use of the CHSH inequality ($n = 2$) is $\frac{1}{\sqrt{2}}$ and is, interestingly, shown to be saturated by quantum mechanics [5]. Using our result every two-partite Bell's inequality which provides an upper bound on the Grothendieck constant $K_G(d)$ for d-dimensional systems and involves n settings at one of the parties (if the numbers are different for different parties, any number can be taken) gives a bound of $\frac{1}{K_G(d)}$ on the shrinking factor of symmetric $1 \longrightarrow n$ cloning machine. For example, the recent result [20] provides us with stronger bounds on the shrinking factors for the symmetric cloning of qubits for cloning machines that make a very large number of copies (at least $1 \longrightarrow 465$).

In conclusion, we derive monogamy constraints on correlations using only nonsignaling condition. Our results can be applied to any Bell's inequality, and in each case, the constraints they give are easy to calculate. These constraints hold for every nonsignaling theory, quantum mechanics being a special case. This generalizes previously known results which either were explicitly obtained for only the CHSH inequality or gave only qualitative description of monogamy. We also have shown an exemplary application of our results in finding a straightforward way to derive bounds on shrinking factors of cloning machines in any nonsignaling theory.

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