"Fast" Nonlinear Evolution in Wave Turbulence

S. Y. Annenkov and V. I. Shrira
Department of Mathematics, EPSAM, Keele University, Keele ST5 5BG, United Kingdom
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Modeling of nonlinear random wave fields in nature (and, in particular, their most common example—wind waves in the ocean) is one of the fundamental open problems of natural sciences. The existing theoretical approaches based on the kinetic equation paradigm assume a proximity to stationarity and homogeneity. In reality this assumption is often violated and how a wave field evolves is not known. We show by direct numerical simulation that after a strong perturbation the wave field evolves on the much faster $O(\varepsilon^{-2})$ "dynamic" [rather then $O(\varepsilon^{-4})$ "kinetic"] time scale; here ε is the characteristic wave steepness ($\varepsilon \ll 1$). The phenomenon of fast evolution is universal, and it must occur whenever there is a strong external perturbation.

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Introduction.—Modeling of random nonlinear wave fields (wave turbulence) is a fundamental problem of great significance. Wave turbulence occurs at all scales, from quantum to cosmological, and in many physical contexts, with applications ranging from oceans and atmospheres to semiconductor lasers and thermonuclear devices. Wind waves represent the most well-studied example of wave turbulence with immediate practical applications. The quality of wave forecasts based upon wave modeling is essential for the reliability of shipping, development of offshore resources and global weather prediction [1]. The reduction of the wave turbulence problem to an analysis of the kinetic equation is one of the major achievements of the 20th century physics [2,3]. The fundamentals of the present understanding of wind-wave evolution are as follows: (i) Wind waves are weakly nonlinear, and their characteristic steepness ε (a measure of nonlinearity) is small [1,4]. (ii) Dominant waves evolve primarily due to nonlinear interactions, rather than due to the direct effect of wind [5]. (iii) Because of the dominant (quartet) nonlinear interactions "individual" wave amplitudes evolve on the $O(\varepsilon^{-2})$ time scale. (iv) Random wind-wave fields are described by ensemble averaged quantities; then the $O(\varepsilon^{-2})$ dynamics of the individual waves averages out and the evolution of wave spectra occurs on the $O(\varepsilon^{-4})$ time scale and is described by the kinetic equation. (v) The kinetic equation prescribes the scaling of energy fluxes as $O(\varepsilon^{6})$, and the existence of Kolmogorov-Zakharov (KZ) cascades of energy and wave action that gives rise to the development of powerlike wave spectra [1,2,4–7]. This picture, supported experimentally [1,4,7] and by numerical simulations [8–10], assumes that the wave fields are close to stationarity, and that there is spatial homogeneity of the environment. A key open question is what happens when these assumptions are violated in nature, and the theory cannot be applied. In particular, although sudden changes of wind speed or direction often occur in real oceanic conditions and violate the applicability conditions of the theory, modeling of such situations is still performed with the kinetic equation, due to the absence of alternatives [4,11,12]. There is experimental evidence that the adjustment of the wave fields to sudden change of wind direction is faster than predicted by the kinetic equation [13,14], and that a sharp change of wind speed can also cause a much faster field evolution [15]. In the most detailed and well-documented laboratory study of effects of a sudden wind increase [16], it was found that the wave field adjustment occurs on the time scale of only a few dozen wave periods, which is incompatible with the $O(\varepsilon^{-4})$ time scale estimate.

On the theoretical side it was first noted that the evolution of an arbitrary initial wave field not in the near-stationary state can occur on the dynamic $O(\varepsilon^{-2})$ time scale during a short initial stage [17]; the conjecture was supported by the simulations of the Dysthe equation [18] and later confirmed by integration of the Zakharov equation [8]. It was also found that a sharp change of forcing entails evolution of spectra on the $O(\varepsilon^{-2})$ time scale at least for toy models describing a few interacting wave packets [19]. Theoretical arguments suggesting that such a fast evolution is a generic phenomenon, whenever there is a strong perturbation, were put forward in [19], but whether this is true for real systems remained an open question. The most natural way to address it is via direct numerical simulations (DNS). However, solving the Navier-Stokes or Euler equations would make little sense because wave breaking is unavoidable and cannot be resolved. For modeling wave nonlinear interactions the integrodifferential Zakharov equation is fundamental, does not have the breaking problem and has the advantage of universality. Here, using a specially developed DNS algorithm for the Zakharov equation [8], we show that fast evolution does occur for realistic forcing, and conclude that it must be a commonplace phenomenon in nature. For wind waves such regimes are commonly caused by, e.g., sudden changes of wind, currents, internal waves, interaction with bodies and shores; modeling of all such situations should be radically revised.

Theoretical background.—We consider potential gravity waves on the surface of deep ideal fluid governed by the Zakharov equation [2,20,21]

$$i\frac{\partial b_0}{\partial t} = (\omega_0 + i\gamma_0)b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3} d\mathbf{k}_{123}. \tag{1}$$

Here, $b(\mathbf{k})$ is a canonical complex variable in Fourier space, $\mathbf{k}$ is the wave vector, $k = |\mathbf{k}|$, $\omega(\mathbf{k}) = (gk)^{1/2}$ is the linear dispersion relation, $i\gamma(\mathbf{k})$ is the small imaginary correction to frequency due to forcing or dissipation, gravity g is normalized to unity, and integration in (1) is performed over the entire $\mathbf{k}$-plane. The compact notation used designates the arguments by indices, e.g., $T_{0123} = T(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, $\delta_{0+1-2-3} = \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$, asterisk means complex conjugation, and t is time. The canonical variable $b(\mathbf{k})$ is linked to the Fourier-transformed primitive physical variables $\zeta(\mathbf{k}, t)$ and $\varphi(\mathbf{k}, t)$ (position of the free surface and the velocity potential at the surface, respectively) through an integral-power series [21]

$$b(\mathbf{k}) = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{\omega(\mathbf{k})}{k}}\zeta(\mathbf{k}) + i\sqrt{\frac{k}{\omega(\mathbf{k})}}\varphi(\mathbf{k})\right\} + O(\varepsilon).$$

Derivation of (1) assumes that wave slopes are $O(\varepsilon)$ small, and includes expansion in powers of ε. Details of the lengthy procedure of derivation of (1), as well as the expression for the kernel T, can be found in [21].

Statistical description of wave fields is usually sought in terms of correlators of $b(\mathbf{k}, t)$. The classical derivation (e.g., [2]) uses (1) as the starting point and leads to the equation

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2 f_{0123} \delta_{0+1-2-3} \Delta\omega d\mathbf{k}_{123} + S_f, \tag{2}$$

where n_0 is the second-order correlator, $\langle b_0^* b_1 \rangle = n_0 \delta_{0-1}$, angular brackets mean ensemble averaging, $f_{0123} = n_2 n_3(n_0 + n_1) - n_0 n_1(n_2 + n_3)$, $\Delta\omega = \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)$, and S_f is the forcing or dissipation term. The kinetic equation (2) describes the evolution of the wave statistical ensemble. Derivation of (2) assumes that a random wave field is close to stationarity, and (2) predicts slow $O(\varepsilon^{-4})$ evolution, regardless of the initial conditions or forcing changes. Thus, if a wave field is instantly driven out of the equilibrium by a perturbation, then (2), strictly speaking, cannot be used. We consider this case by solving (1) numerically by DNS.

Numerical algorithm.—Direct numerical simulations are based on an efficient algorithm for the solution of (1), first developed for discrete wave systems [20], then modified for the study of continuous random water wave fields and tested to give good agreement with the kinetic equation where it is applicable [8]. We build in Fourier space a grid consisting of a moderate (about 5000) number of wave packets, coupled through exact and approximate resonant interactions. The grid is logarithmic in the wave number k (161 points within a span $0.13 < k < 2.12m^{-1}$) and regular in the angle θ (31 point within $-\pi/3 \leq \theta \leq \pi/3$). A quartet of grid points is assumed to be in approximate resonance if its wave number and frequency mismatch satisfies a pair of conditions $\Delta\omega/\omega_{\min} < \lambda_\omega$, $\Delta k/k_{\min} < \lambda_k \bar{\omega}/\omega_{\min}$, where $\Delta\omega$ and Δk are the frequency and wave number mismatch in the quartet, $\omega_{\min}$ and $k_{\min}$ are the minimum values of frequency and wave number in the quartet, $\bar{\omega}$ is the mean frequency, and λ_ω and λ_k are detuning parameters, chosen to ensure that the total number of resonances is $O(N^2)$, where N is the number of grid points. In this study, $N = 4991$, $\lambda_\omega = \lambda_k = 0.01$; results were verified to be nondependent on specific values in a wide range of λ_ω, λ_k. Forcing is calculated using an empirical formula [22] and confined to the range $1.0 < k < 1.29m^{-1}$, and strong dissipation is applied to $k > 1.62m^{-1}$. The averaging is over 30 realizations.

In modeling the effect of sharp wind change on random wave fields, we focus on wave nonlinear interactions and do not aim to reproduce all the complexity of wave generation and dissipation dominated by breaking. Instead, preserving the basic physics, we choose a simplified description: wave breaking is not explicitly included, and the forcing is wind input minus dissipation. Forcing is applied to the high-frequency part of the spectrum only, which means that for dominant waves and waves of comparable scales, wind input is exactly balanced by dissipation. Then, the most energetic part of wave field evolves entirely due to nonlinear interactions, which is typical of developed sea waves [7,23]. In order to quantify the dependence of wave evolution on nonlinearity, a wide range of wave steepness is used.

Results.—In all numerical runs, we start with low-intensity white noise, and then run the model for several thousand characteristic wave periods, to allow the spectrum to develop under the constant wind. Thus, prior to the change of forcing, the developed wave spectrum is nearly stationary, and its evolution is described by large-time asymptotics of the kinetic equation [5]. In particular, the wave action spectral density of the dominant waves, $N(k_p)$, where k_p is the wave number of the spectral peak, grows slowly: $N(k_p) \sim t^{23/11}$, the peak position gradually shifts to low wave numbers as $k_p \sim t^{-6/11}$ due to the inverse Kolmogorov-Zakharov cascade of wave action, and the dominant wave steepness ε (defined as $\varepsilon = \sqrt{2E}k_p/2\pi$, where $E = \int \omega b_k b_k^* dk$ is the wave energy) slowly decreases. Several values of wind speed in the range 8–16 m/s were used in different runs. In all runs, evolution of wave spectra continued until the spectral peak, which was slowly moving towards large scales, reached the value $k_p \approx 0.45k_f$, where $k_f = 1.0$ is the lower boundary of forcing. Then, at time T_0 (Table I), the wind was instantaneously increased to 16 or 24 m/s, so that the spectrum started to grow from the quasiequilibrium value $N_0(k)$. An example of the evolution of wave action spectrum for run 3

TABLE I. Runs of the numerical model. U_1 and U_2—wind speeds before and after the change of forcing at time T_0; ε_1 and ε_2—equilibrium values of wave steepness; k_p—wave number of the spectral peak at the change of forcing.

Run	U_1 (m/s)	U_2 (m/s)	ε_1	ε_2	T_0 (periods)	k_p (m^{-1})
1	8	16	0.099	0.174	15280	0.45
2	8	24	0.099	0.238	15280	0.45
3	10	16	0.118	0.173	6455	0.51
4	10	24	0.119	0.231	6455	0.51
5	12	16	0.139	0.169	5348	0.46
6	12	24	0.138	0.233	5348	0.46
7	16	24	0.170	0.217	4011	0.40

of the numerical model (Table I) is shown in Fig. 1(a), and the corresponding evolution of wave steepness and spectral peak in Figs. 1(b) and 1(c) respectively. After the sharp increase in wind, the steepness rises almost instantly, and reaches its new "equilibrium value" in a few hundred wave periods. During this short period the disturbance "propagates" from the forcing region to the spectral peak on a time scale of dozens of wave periods, so that energy at all wave numbers on the high-frequency side of the peak starts to grow almost simultaneously, leading to a fast increase in the steepness. Meanwhile, the downshift of the spectral peak stops for a few hundred wave periods, and then resumes at an increased rate, gradually slowing down back to the kinetic equation asymptotics [Fig. 1(c)].

Our primary interest is in growth of waves on the spectral slope, i.e., between the peak and forcing region, during the adjustment process. The response of several sample spectral bands to the wind change is illustrated in Fig. 2(a), shown as function of time in terms of the normalized wave action $(N(k) - N_0(k))/(N_f(k) - N_0(k))$, where $N(k)$ is the wave action at time t, and $N_0(k)$ and $N_f(k)$ are the quasiequilibrium values of the wave action before and after the wind increase. For all spectral bands the growth rates have a peak in the first few hundred wave periods after T_0, and then gradually decrease, as the spectrum slowly adjusts to the new forcing.

We introduce two characteristic values of the growth rate: the "maximal" rate $M(k)$ [defined as the maximum, for each k, of $dN(k)/dt$ over 1000 wave periods after the change of wind], and the "average" rate $m(k)$ defined as the mean growth rate during 3/4 of the total change between $N_0(k)$ and $N_f(k)$. The dependence of $M(k)$ and $m(k)$ on k is shown in Fig. 2(b). There is no net forcing or dissipation for $k < 1$, so that all spectral growth illustrated in Fig. 2 is entirely due to nonlinear interactions. Dependence of the growth rate on time and wave number resembles the results of simulations of the kinetic equation [11]; however, we are primarily interested in its dependence on nonlinearity ε (the kinetic equation predicts strict ε^6 scaling). Here, nonlinearity can be characterized by two parameters, namely, the "equilibrium" values of ε before and after the wind change (ε_1 and ε_2 respectively, see Table I). In all runs, dependence of $m(k)$ on ε_1 was weak, and virtually no dependence of $M(k)$ on ε_1 was found, but both $m(k)$ and $M(k)$ strongly depend on ε_2. To quantify this dependence, we choose the integrals of $m(k)$ and $M(k)$ with respect to k on the spectral slope as characteristic values, denote them as $\bar{m}$ and $\bar{M}$, and plot them for all runs versus ε_2, on a logarithmic scale (Fig. 3). Both $\bar{m}$ and $\bar{M}$ are scaled with ε^λ, where λ is close to 4 (least-squares fits yield 4.3 and 3.5). This scaling is in sharp contrast with the ε^6 scaling of the kinetic equation, and corresponds to the $O(\varepsilon^{-2})$ evolution time scale.

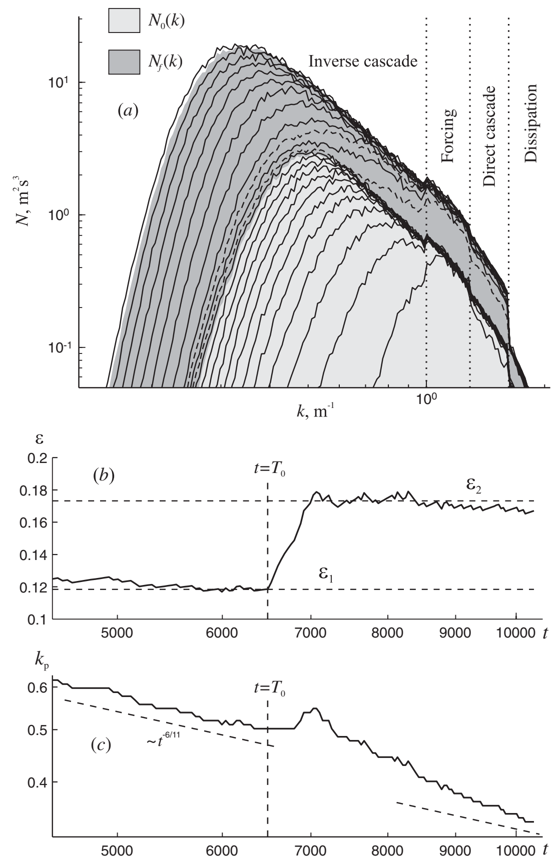

FIG. 1. An example of DNS of random wave field evolution under an abrupt increase of wind (Run 3). (a) Time evolution of wave action spectrum, in steps of approximately 400 periods of the spectral peak (intermediate curves are dashed). Regions of forcing and dissipation are delimited by dotted vertical lines. Equilibrium spectra $N_0(k)$ and $N_f(k)$ are shown by shading. (b) Evolution of wave steepness ε. Equilibrium values of steepness before and after the increase of forcing (which occurs at $t = T_0$) are marked by dashed horizontal lines. (c) Evolution of spectral peak k_p. Theoretical (according to the large-time kinetic equation asymptotics) downshift rate $k_p \sim t^{-6/11}$ is shown by the dashed line.

Conclusions.—The DNS of wave fields subjected to an abrupt change of forcing shows that the nonlinear evolution of random wave fields occurs on the fast (dynamic) time scale. This is similar to the fast evolution of model spectra at a short initial stage found earlier by two different meth-

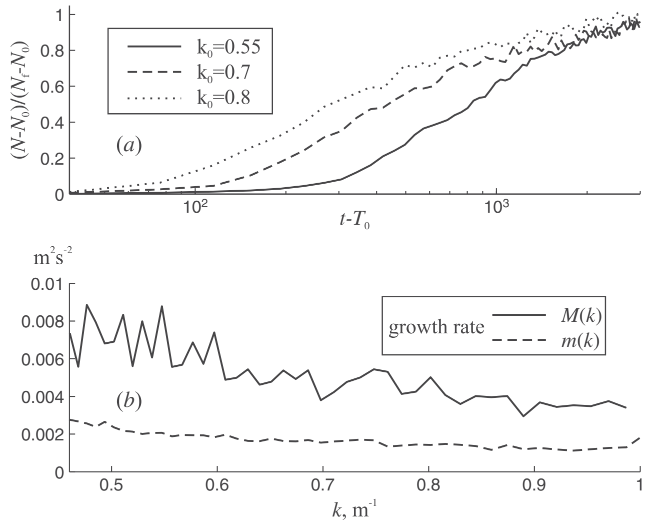

FIG. 2. Response of the wave action spectrum to the sharp increase of forcing (run 3). (a) Response of sample spectral bands (integrated from k_0 to $1.1k_0$). Values of the integrated spectrum $N(k)$ are normalized by the equilibrium values $N_0(k)$ and $N_f(k)$. (b) Maximal and average growth rates ($M(k)$ and $m(k)$ respectively, defined in the text) as functions of the wave number.

ods [8,18], since the initialization of a spectrum by random initial phases is another example of a strong perturbation. This implies that wave modeling in common situations of rapidly changing or gusty winds, or in the presence of spatial inhomogeneities, should be radically revised. Since this study is based upon the Zakharov equation, in which all specific properties of waves are contained merely

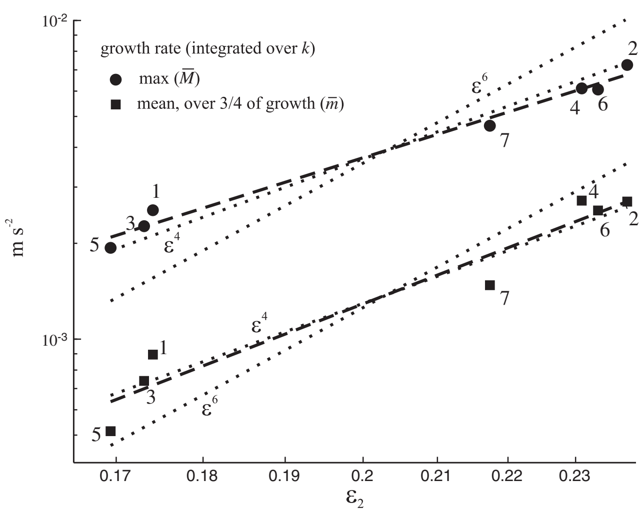

FIG. 3. Integrated (from $k = 0.5$ to $k = 1.0$) maximal growth rates $\bar{M}$ and mean growth rates $\bar{m}$ versus wave steepness ε_2, on a logarithmic scale, with least-squares fits (dashed lines) for the scaling ε^λ ($\lambda = 3.5$ for $\bar{M}$ and $\lambda = 4.3$ for $\bar{m}$). Theoretical dynamic ε^4 and kinetic ε^6 scalings are also shown (dotted lines). Numbers identify numerical runs (Table I).

in the interaction coefficients [2], this conclusion about the scaling of wave field evolution is not confined to wind waves, but applies to all random weakly nonlinear dispersive wave fields whenever there is a sharp and strong external perturbation.

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