

## Quasilocal Mass in General Relativity

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There have been many attempts to define the notion of quasilocal mass for a spacelike two surface in spacetime by the Hamilton-Jacobi analysis. The essential difficulty in this approach is to identify the right choice of the background configuration to be subtracted from the physical Hamiltonian. Quasilocal mass should be non-negative for surfaces in general spacetime and zero for surfaces in flat spacetime. In this Letter, we propose a new definition of gauge-independent quasilocal mass and prove that it has the desired properties.

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*Introduction.*—As is well known from the equivalence principle, there is no well-defined concept of energy density in general relativity. On the other hand, when there is asymptotic symmetry, concepts of total energy and momentum can be defined. This is called the ADM energy-momentum and the Bondi energy-momentum when the system is viewed from spatial infinity and null infinity, respectively. These concepts are fundamental in general relativity and have been proven to be natural and to satisfy the important positivity condition in the work of Schoen-Yau [1], Witten [2], etc. However, there are limitations to such definitions if the physical system is not isolated and cannot quite be viewed from infinity where asymptotic symmetry exists. It was proposed more than 40 years ago to measure the energy of a system by enclosing it with a membrane, namely, a closed spacelike two surface, and then attach to it an energy-momentum four vector. It is natural to expect that the four vector will depend only on the induced metric, the second fundamental form, and the connection on the normal bundle of the surface embedded in spacetime. This is the idea behind the definition of quasilocal mass of this surface. Obviously, there are a few conditions the quasilocal mass has to satisfy. First, the ADM or Bondi mass should be recovered as spatial or null infinity is approached. Second, the correct limits need be obtained when the surface converges to a point. Third and most importantly, quasilocal mass must be non-negative in general and zero when the ambient spacetime of the surface is the flat Minkowski spacetime. It should also behave well when the spacetime is spherically symmetric. Many proposals were made by Hawking [3], Penrose [4], etc. The most promising one was proposed by Brown and York [5] where they motivated their definition by using the Hamiltonian formulation of general relativity (see also Hawking-Horowitz [6]). They found interesting local quantities from which the definition of quasilocal mass was extracted. Their definition depends on

the choice of gauge along the three-dimensional spacelike slice which the surface bounds. It has the right asymptotic behavior but is not positive in general. Shi-Tam [7] proved that it is positive when the three-dimensional slice is time symmetric. Motivated by geometric consideration, Liu and Yau [8] (see also Kijowski [9], Booth-Mann [10], and Epp [11]) introduced a mass which is gauge independent, and proved that it is always positive. However, it was pointed out by ÓMurchadha *et al.* [12] that the Liu and Yau mass can be strictly positive even when the surface is in a flat spacetime. In this Letter, we explore more in the direction of the Hamilton-Jacobi analysis of Brown and York. Combining some ideas of Liu and Yau, we define a quasilocal mass which is gauge independent and non-negative. Moreover, it is zero whenever the surface is in the flat Minkowski spacetime. We believe that the present definition satisfies all the requirements necessary for a valid definition of quasilocal mass, and it is likely to be the unique definition that satisfies all the desired properties.

*Hamiltonian formulation revisited.*—Consider a spacetime region  $M$  that is foliated by a family of spacelike hypersurface  $\Omega_t$  for  $t$  in the time interval  $[t', t'']$ . The boundary of  $M$  consists of  $\Omega_{t'}$ ,  $\Omega_{t''}$ , and  ${}^3B$ . Let  $u^\mu$  denote the future-pointing timelike unit normal to  $\Omega_t$ . Assume  $u^\mu$  is tangent to  ${}^3B$ . Denote the boundary of  $\Omega_t$  by  $\Sigma_t$  which is the intersection of  $\Omega_t$  and  ${}^3B$ . Let  $v^\mu$  denote the outward pointing spacelike unit normal of  $\Sigma_t$  such that  $u_\mu v^\mu = 0$ . Denote by  $k$  the trace of the two-dimensional extrinsic curvature of  $\Sigma_t$  in  $\Omega_t$  in the direction of  $v^\mu$ . Denote the Riemannian metric, the extrinsic curvature, and the trace of the extrinsic curvature on  $\Omega_t$  by  $g_{\mu\nu}$ ,  $K_{\mu\nu} = \nabla_\mu u_\nu$ , and  $K = g^{\mu\nu} K_{\mu\nu}$ , respectively. Let  $t^\mu$  be a timelike vector field satisfying  $t^\mu \nabla_\mu t = 1$ .  $t^\mu$  can be decomposed into the lapse function and shift vector  $t^\nu = Nu^\nu + N^\nu$ . Let  $S$  denote the action for  $M$ , then the Hamiltonian at  $t''$  is given by  $H = -\frac{\partial S}{\partial t''}$ . The calculation in Brown and York [5] (see also Hawking-Horowitz [6]) leads to

$$H = -\frac{1}{8\pi} \int_{\Sigma, \mu} [Nk - N^\mu v^\nu (K_{\mu\nu} - K g_{\mu\nu})] \quad (1)$$

on a solution  $M$  of the Einstein equation. To define quasilocal energy, we need to find a reference action  $S_0$  that corresponds to fixing the metric on  ${}^3B$  and compute the corresponding reference Hamiltonian  $H_0$ . The energy is then  $E = -\frac{\partial}{\partial t^\nu} (S - S_0) = H - H_0$ . In Brown and York's prescription, the reference is taken to be an isometric embedding of  $\Sigma$  into  $\mathbb{R}^3$ , considered as a flat three-dimensional slice with  $K_{\mu\nu} = 0$  in a flat spacetime. Choosing  $N = 1$  and  $N^\mu = 0$ , the Brown and York quasilocal energy is  $\frac{1}{8\pi} \int_{\Sigma} (k_0 - k)$ . References such as surfaces in the light cones ([13,14]) and other conditions [9] have been proposed. However, the Brown and York energy for the examples [12] of surfaces in the Minkowski space are in general nonzero for all these references. We shall define quasilocal energy using general isometric embeddings into  $\mathbb{R}^{3,1}$  as reference configurations. In particular, the  $t^\nu$  is obtained by transplanting a Killing vector field in  $\mathbb{R}^{3,1}$  to the physical spacetime through the embedding.

*Definition of quasilocal energy in the canonical gauge.*—Suppose  $\Sigma$  is a spacelike surface in a time-orientable spacetime,  $u^\nu$  is a future-pointing timelike unit normal, and  $v^\nu$  is a spacelike unit normal with  $u^\nu v_\nu = 0$  along  $\Sigma$ . We assume  $v^\nu$  is the outward normal of a spacelike hypersurface  $\Omega$  that is defined locally near  $\Sigma$ . Consider the 4-vector field

$$ku^\nu + v^\mu (K_\mu^\nu - K \delta_\mu^\nu) \quad (2)$$

along  $\Sigma$ . The definition of this vector field only depends on the two normals  $u^\nu$  and  $v^\nu$  along  $\Sigma$ . The normal component (with respect to  $\Sigma$ ) of (2) is  $j^\nu = ku^\nu - pv^\nu$ , where  $p = K - K_{\mu\nu} v^\mu v^\nu$ .  $j^\nu$ , as well as the mean curvature vector field,  $h^\nu = -kv^\nu + pu^\nu$ , are defined independent of the choice of gauge  $u^\nu$  and  $v^\nu$ .

Consider a reference isometric embedding  $i: \Sigma \hookrightarrow \mathbb{R}^{3,1}$  of  $\Sigma$ . Fix a constant timelike unit vector  $t_0^\nu$  in  $\mathbb{R}^{3,1}$ , and choose a preferred pair of normals  $u_0^\nu$  and  $v_0^\nu$  along  $i(\Sigma)$  in the following way: Take a spacelike hypersurface  $\Omega_0$  with  $\partial\Omega_0 = i(\Sigma)$  and such that the outward pointing spacelike unit normal  $v_0^\nu$  of  $\partial\Omega_0$  satisfies  $(t_0)_\nu v_0^\nu = 0$ . Let  $u_0^\nu$  be the future-pointing timelike unit normal of  $\Omega_0$  along  $i(\Sigma)$ . We can similarly form  $k_0 u_0^\nu + v_0^\mu [(K_0)_\mu^\nu - K_0 \delta_\mu^\nu]$  in terms of the corresponding geometric quantities on  $\Omega_0$  and  $i(\Sigma)$ .  $(u_0^\nu, v_0^\nu)$  along  $i(\Sigma)$  in  $\mathbb{R}^{3,1}$  is the reference normal gauge we shall fix, and it depends on the choice of the pair  $(i, t_0^\nu)$ .

When the mean curvature vector  $h^\nu$  of  $\Sigma$  in  $M$  is spacelike, a reference isometric embedding  $i: \Sigma \hookrightarrow \mathbb{R}^{3,1}$  and  $t_0^\nu \in \mathbb{R}^{3,1}$  determine a canonical future-pointing timelike normal vector field  $\bar{u}^\nu$  in  $M$  along  $\Sigma$ . Indeed, there is a unique  $\bar{u}^\nu$  that satisfies

$$h_\nu \bar{u}^\nu = (h_0)_\nu u_0^\nu \quad (3)$$

where  $h_0^\nu$  is the mean curvature vector of  $i(\Sigma)$  in  $\mathbb{R}^{3,1}$ .

Physically, (3) means the expansions of  $\Sigma \subset M$  and  $i(\Sigma) \subset \mathbb{R}^{3,1}$  along the respective directions  $\bar{u}^\nu$  and  $u_0^\nu$  are the same. This condition corresponds to fixing the metric on  ${}^3B$  up to the first order in choosing the reference Hamiltonian.  $\bar{u}^\nu$  shall be called the *canonical gauge* with respect to the pair  $(i, t_0^\nu)$ . Take  $\bar{v}^\nu$  to be the spacelike normal vector that is orthogonal to  $\bar{u}^\nu$  and satisfies  $\bar{v}^\nu h_\nu < 0$ , and take a spacelike hypersurface  $\bar{\Omega}$  in  $M$  such that  $\bar{v}^\nu$  is the outward normal. We can similarly form  $\bar{k} \bar{u}^\nu + \bar{v}^\mu (\bar{K}_\mu^\nu - \bar{K} \delta_\mu^\nu)$ , where  $\bar{K}_\mu^\nu, \bar{K}$ , and  $\bar{k}$  are the corresponding data on  $\bar{\Omega}$ . The trace of the two-dimensional extrinsic curvature  $\bar{k}$  of  $\Sigma$  with respect to  $\bar{v}^\nu$  is then given by  $\bar{k} = -\bar{v}^\nu h_\nu > 0$ .

Four vectors in  $\mathbb{R}^{3,1}$  and  $M$ , along  $i(\Sigma)$  and  $\Sigma$ , respectively, can be identified through

$$u_0^\nu \rightarrow \bar{u}^\nu, \quad v_0^\nu \rightarrow \bar{v}^\nu, \quad (4)$$

and the identification of tangent vectors on  $i(\Sigma)$  and  $\Sigma$ .

The quasilocal energy of  $\Sigma$  in the canonical gauge with respect to  $(i, t_0^\nu)$  is defined to be

$$\frac{1}{8\pi} \int_{\Sigma} \{ \bar{k} \bar{u}^\nu + \bar{v}^\mu (\bar{K}_\mu^\nu - \bar{K} \delta_\mu^\nu) - k_0 u_0^\nu - v_0^\mu [(K_0)_\mu^\nu - K_0 \delta_\mu^\nu] \} (t_0)_\nu, \quad (5)$$

where  $\bar{u}^\nu$  is determined by (3), the identification (4) is used, and  $t_0^\nu$  in  $\mathbb{R}^{3,1}$  is identified with  $N_0 \bar{u}^\nu + N_0^\nu$  in  $M$ . In terms of the lapse  $N_0$  and shift  $N_0^\nu$ , the quasilocal energy is

$$\frac{1}{8\pi} \int_{\Sigma} (k_0 - \bar{k}) N_0 - [v_0^\mu (K_0)_{\mu\nu} - \bar{v}^\mu \bar{K}_{\mu\nu}] N_0^\nu. \quad (6)$$

The mean curvature vector  $h^\nu$  being spacelike is equivalent to  $\rho\mu > 0$  where  $\rho$  and  $\mu$  are the expansion along the future and past outer null normals of  $\Sigma$ . When the image of the reference embedding lies in a flat space slice in  $\mathbb{R}^{3,1}$ , we have  $t_0^\nu = u_0^\nu$ . On the other hand, the canonical gauge is  $\bar{u}^\nu = \frac{1}{\sqrt{8\rho\mu}} j^\nu$ , and we derive that  $\bar{k} = \sqrt{8\rho\mu}$ . In this case, (5) recovers the Liu and Yau quasilocal mass  $\frac{1}{8\pi} \int_{\Sigma} (k_0 - \sqrt{8\rho\mu})$ .

*Admissible pairs.*—Unlike Brown and York or Liu and Yau, we do not require the surface  $\Sigma$  to have positive Gauss (intrinsic) curvature and apply the embedding theorem of Weyl. Instead, we prove a uniqueness and existence theorem of isometric embeddings into the Minkowski space under a more general convexity condition.

*Definition 1.*—Let  $t_0^\nu$  be a constant timelike unit vector in  $\mathbb{R}^{3,1}$ . A closed surface  $\Sigma$  in  $\mathbb{R}^{3,1}$  is said to have convex shadow in the direction of  $t_0^\nu$  if the projection of  $\Sigma$  onto the orthogonal complement  $\mathbb{R}^3$  of  $t_0^\nu$  is a convex surface.

The set of isometric embeddings with convex shadows is parametrized by functions satisfying a convexity condition.

*Theorem 1.*—Let  $\sigma_{ab}$  be a Riemannian metric on a two-sphere  $\Sigma$ . Given any function  $\tau$  on  $\Sigma$  with

$$\kappa + (1 + \sigma^{ab} \nabla'_a \tau \nabla'_b \tau)^{-1} \frac{\det(\nabla'_a \nabla'_b \tau)}{\det \sigma_{ab}} > 0 \quad (7)$$

where  $\kappa$  is the Gauss curvature and  $\nabla'$  is the covariant derivative of the metric  $\sigma_{ab}$ . Then, there exists a unique spacelike embedding  $i: \Sigma \hookrightarrow \mathbb{R}^{3,1}$  such that the time function restricts to  $\tau$  on  $i(\Sigma)$ , and the induced metric on  $i(\Sigma)$  is  $\sigma_{ab}$ .

Uniqueness is dealt with first. Suppose there are two such isometric embeddings  $i_1$  and  $i_2$  with the same time function  $\tau$ . It is not hard to check that the condition (7) implies the projections of  $i_1(\Sigma)$  and  $i_2(\Sigma)$  onto the orthogonal complement of the time direction are isometric as convex surfaces in  $\mathbb{R}^3$ . By Cohn-Vossen rigidity theorem, the projections are congruent by a rigid motion of  $\mathbb{R}^3$ . Since they have the same time functions,  $i_1(\Sigma)$  and  $i_2(\Sigma)$  are congruent by a Lorentzian rigid motion of  $\mathbb{R}^{3,1}$ . To prove existence, condition (7) is shown to imply that the metric  $\sigma_{ab} + \nabla'_a \tau \nabla'_b \tau$  has positive Gauss curvature and thus can be isometrically embedded into  $\mathbb{R}^3$ . We may assume this  $\mathbb{R}^3$  is a space slice in the Minkowski space, so the induced metric on the graph of  $\tau$  in  $\mathbb{R}^{3,1}$  is exactly  $\sigma_{ab}$ . This completes the proof of Theorem 1.

In order to recognize a surface in the Minkowski space, we solve a Dirichlet boundary value problem for Jang's equation. Given a hypersurface  $(\Omega, g_{ij}, K_{ij})$  in  $M$ , Jang's equation seeks for a solution  $f$  of

$$\sum_{i,j=1}^3 \left( g^{ij} - \frac{D^i f D^j f}{1 + g^{ij} D_i f D_j f} \right) \times \left( \frac{D_i D_j f}{(1 + g^{ij} D_i f D_j f)^{1/2}} - K_{ij} \right) = 0, \quad (8)$$

where  $D$  is the covariant derivative of  $g_{ij}$ . The graph of  $f$  in the space  $\Omega \times \mathbb{R}$  is denoted by  $\tilde{\Omega}$ . In this Letter, we are interested in the case when  $\partial\Omega = \Sigma$  and the prescribed value of  $f$  on the boundary  $\Sigma$  is given. Notice that if  $\Sigma$  is in  $\mathbb{R}^{3,1}$  and if we take the time function  $\tau$  as the boundary value to solve Jang's equation, then  $\tilde{\Omega}$  will be a flat domain in  $\mathbb{R}^3$ .

**Definition 2.**—Consider a spacelike two surface  $\Sigma$  in a time-orientable spacetime  $M$ , an isometric embedding  $i: \Sigma \hookrightarrow \mathbb{R}^{3,1}$ , and a constant timelike unit vector  $t_0^\nu \in \mathbb{R}^{3,1}$ . Let  $\tau$  denote the time function restricted to  $i(\Sigma)$ .  $(i, t_0^\nu)$  is said to be an admissible pair for  $\Sigma$  if the following conditions are satisfied: (A)  $i$  has convex shadow in the direction of  $t_0^\nu$ . (B)  $\Sigma$  bounds a spacelike domain  $\Omega$  in  $M$  such that Jang's Eq. (8) with the Dirichlet boundary data  $\tau$  is solvable on  $\Omega$  (with possible apparent horizons in the interior). (C) Suppose  $f$  is the solution of Jang's equation in (B) and  $v^\nu$  is the outward unit normal of  $\Sigma$  that is tangent to  $\Omega$ , and  $u^\nu$  is the future-pointing timelike normal of  $\Omega$  in  $M$ . Consider the new gauge given by  $u'^\nu = \sinh\phi v^\nu + \cosh\phi u^\nu$  and  $v'^\nu = \cosh\phi v^\nu + \sinh\phi u^\nu$ , where  $\sinh\phi =$

$\frac{f_\nu}{\sqrt{1 + \sigma^{ab} \nabla'_a \tau \nabla'_b \tau}}$ , and  $f_\nu$  is the normal derivative of  $f$  in the direction  $v^\nu$ . We require that

$$k' N_0 - N_0^\nu v'^\mu K'_{\mu\nu} > 0, \quad (9)$$

where  $k'$ ,  $g'_{\mu\nu}$ , and  $K'_{\mu\nu}$  are the corresponding data on the new three dimensional spacelike domain  $\Omega'$  spanned by  $v'^\nu$ , and  $N_0$  and  $N_0^\nu$  are the lapse function and shift vector of  $t_0^\nu = N_0 u_0^\nu + N_0^\nu$ .

*Remark 1.*—By a barrier argument, we show that  $\Omega$  satisfies (B) if on  $\Sigma$ ,  $k > \frac{1}{t^a t_a (1 + t^a t_a)} (K_{ab} t^a t^b) + K_{ab} u^a u^b$  where  $u^a$  is a two-vector such that  $t^a u_a = 0$  and  $u^a u_a = 1$ , and  $t^a = \pi_v^a t_0^\nu$  is the projection of  $t_0^\nu$  onto  $\Sigma$ . Also by elliptic estimates, (C) will be satisfied if (9) holds for  $u^\nu$  and  $v^\nu$ , and  $\sigma^{ab} \nabla'_a \tau \nabla'_b \tau$  is small enough. In particular, if  $\Sigma$  has positive Gauss curvature and spacelike mean curvature vector in  $M$ , then any isometric embedding  $i$  whose image lies in an  $\mathbb{R}^3$  is admissible. Conditions (B) and (C) are necessary for the present proof of the positivity result Theorem 2, and they are general enough to include this most important case. We expect the most general positivity result holds under a convexity condition involving the Hamiltonian surface density four vector (2) of  $\Sigma$  and the function  $\tau$ .

*Positivity of quasilocal energy.*—We emphasize that although the definition of admissible pairs involves solving Jang's equation, our results only depend on the solvability and not on the specific solution. The expression of quasilocal energy only depends on the canonical gauge  $\bar{u}^\nu$ .

*Theorem 2.*—Suppose  $M$  is a time-orientable spacetime that satisfies the dominant energy condition. Suppose  $\Sigma$  has spacelike mean curvature vector in  $M$ . Then, the quasilocal energy (5) with respect to any admissible pair  $(i, t_0^\nu)$  is non-negative.

We take the time function  $\tau$  on  $i(\Sigma)$  and consider the Dirichlet problem of Jang's equation (8) over  $(\Omega, g_{ij}, K_{ij})$  with  $f = \tau$  on  $\Sigma$ . Condition (B) guarantees the equation is solvable on  $\Omega$ . Denote by  $\tilde{\Omega}$  the graph of the solution of Jang's equation. Schoen and Yau [15] showed that if  $M$  satisfies the dominant energy condition, there exists a vector field  $X$  on  $\tilde{\Omega}$  such that

$$R \geq 2|X|^2 - 2\text{div}X \quad (10)$$

where  $R$  is the scalar curvature of  $\tilde{\Omega}$ .

Let  $\tilde{\Sigma}$  be the graph of  $\tau$  over  $\Sigma$ , and denote the outward normal of  $\tilde{\Sigma}$  with respect to  $\tilde{\Omega}$  by  $\tilde{v}^i$  and the mean curvature of  $\tilde{\Sigma}$  with respect to  $\tilde{v}^i$  by  $\tilde{k}$ . The boundary calculation in [16] shows that  $\sqrt{1 + |\nabla\tau|^2} (\tilde{k} - \tilde{v}^i X_i)$  is equal to the expression in (9), and thus Condition (C) guarantees  $\tilde{k} - \tilde{v}^i X_i > 0$ . We make use of another important property of the canonical gauge that

$$\int_{\tilde{\Sigma}} (\tilde{k} - \tilde{v}^i X_i) \geq - \int_{\tilde{\Sigma}} [\tilde{k} \tilde{u}^\nu + \tilde{v}^\mu (\tilde{K}_\mu^\nu - \tilde{K} \delta_\mu^\nu)] (t_0)_\nu. \quad (11)$$

This is why the eventual definition of the quasilocal energy is independent of the solution of Jang's equation. On the other hand, it is not hard to check that  $-\int_{\tilde{\Sigma}} [k_0 u_0^\nu + v_0^\mu (K_{0\mu}^\nu - K_0 \delta_\mu^\nu)] (t_0)_\nu = \int_{\tilde{\Sigma}} \hat{k}$ . (We were motivated by Gibbon's paper [17] to study this expression.) Here,  $\hat{\Sigma}$  is the image of the projection of  $i(\Sigma)$  onto the orthogonal complement of  $t_0^\nu$ , and  $\hat{k}$  is the mean curvature of  $\hat{\Sigma}$ . Therefore, the proof is reduced to the inequality  $\int_{\tilde{\Sigma}} \hat{k} \geq \int_{\tilde{\Sigma}} (\tilde{k} - \tilde{v}^i X_i)$ . We note that the Riemannian metrics on  $\tilde{\Sigma}$  and  $\hat{\Sigma}$  are the same. The proof will be completed by the following comparison theorem in [16] for the solution of Jang's equation. Suppose  $\tilde{\Omega}$  is a Riemannian three manifold with boundary  $\tilde{\Sigma}$ , and suppose there exists a vector field  $X$  on  $\tilde{\Omega}$  such that (10) holds on  $\tilde{\Omega}$  and  $\tilde{k} > \tilde{v}^i X_i$  on  $\tilde{\Sigma}$ . If the Gauss curvature of  $\tilde{\Sigma}$  is positive, and  $k_0$  is the mean curvature of the isometric embedding of  $\tilde{\Sigma}$  into  $\mathbb{R}^3$ , then  $\int_{\tilde{\Sigma}} k_0 \geq \int_{\tilde{\Sigma}} (\tilde{k} - \tilde{v}^i X_i)$ .

*Definition of quasilocal mass and its positivity.*—Our definition of quasilocal mass is similar to recovering the rest mass of a particle from the energy as measured by all observers of unit four velocities. The *quasilocal mass* of  $\Sigma$  in  $M$  is defined to be the infimum of the quasilocal energy (5) among all admissible pairs  $(i, t_0^\nu)$ .

Under the assumptions of Theorem 2, we obtain

*Theorem 3.*—If the set of admissible pairs is nonempty, then the quasilocal mass of  $\Sigma$  in  $M$  is non-negative. In particular, this is the case when  $\Sigma$  has positive Gauss curvature.

The first part is clear from the definition. When  $\Sigma$  has positive Gauss curvature, we can use Weyl's isometric embedding theorem to embed  $\Sigma$  into a flat space slice  $\mathbb{R}^3$  on which the time function in  $\mathbb{R}^{3,1}$  is a constant. Thus, the admissible set is nonempty by Remark 1. This completes the proof of Theorem 3.

Suppose the infimum is achieved by an admissible pair  $(i, t_0^\nu)$ , the quasilocal energy-momentum four vector is defined as  $m(\Sigma)t_0^\nu$  where  $m(\Sigma)$  is the quasilocal mass of  $\Sigma$ . Therefore, Theorem 3 implies the quasilocal energy-momentum four vector is always future pointing and non-spacelike whenever it is defined.

*Properties of the new quasilocal mass.*—Expression (5) contains the desired correction term; so the examples of surfaces in  $\mathbb{R}^{3,1}$  found in [12] have zero quasilocal mass. The new quasilocal mass given in the previous section has the following properties: (i) Suppose  $\Sigma$  is a spacelike two surface which bounds a spacelike hypersurface  $\Omega$  in a

spacetime  $M$ . The quasilocal mass is defined when the mean curvature vector of  $\Sigma$  in  $M$  is spacelike and the definition is independent of the choice of  $\Omega$ . If  $M$  satisfies the dominant energy condition and  $\Sigma$  has positive intrinsic curvature, then the quasilocal mass is non-negative. More generally, this holds if the set of admissible pairs for  $\Sigma$  is nonempty. (ii) Any spacelike two surface in  $\mathbb{R}^{3,1}$  with convex shadow in a time-direction (see Definition 1) has zero quasilocal mass. (iii) The small sphere limits of the quasilocal mass recover the matter energy-momentum tensor in the presence of matter and the Bel-Robinson tensor *in vacuo*, and the large sphere limits approach the ADM mass in the asymptotically flat case and the Bondi mass in the asymptotically null case.

We remark that the admissible pairs form an open subset of the set of functions  $\tau$  on  $\Sigma$  that satisfies (7). The condition that the admissible set is nonempty in Theorem 3 is a very mild assumption, and the quasilocal mass should be positive regardless of the sign of the intrinsic curvature of  $\Sigma$ . The Euler-Lagrange equation for the energy minimizing isometric embedding into  $\mathbb{R}^{3,1}$  is derived in [16]. When a  $\Sigma$  in spacetime is given, we can solve for this equation and define the quasilocal energy-momentum four vector. The monotonicity property of our mass will be discussed in a forthcoming paper.

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