Creation and Detection of a Mesoscopic Gas in a Nonlocal Quantum Superposition

Christoph Weiss and Yvan Castin
Laboratoire Kastler Brossel, École Normale Supérieure, UPMC and CNRS, 24 rue Lhomond, 75231 Paris Cedex 05, France
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We investigate the scattering of a quantum matter wave soliton on a barrier in a one-dimensional geometry, and we show that it can lead to mesoscopic quantum superposition states, where the atomic gas is in a coherent superposition of being in the half-space to the left of the barrier and being in the half-space to the right of the barrier. We propose an interferometric method to reveal the coherent nature of this superposition, and we discuss in detail the experimental feasibility.

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It is now possible to control the strength of the atomic interaction in a gas, with Feshbach resonances. This has allowed the observation of single matter wave bright solitons with thousands of atoms [1] or a train of solitons [2] with ^{7}Li atoms trapped in a one-dimensional (1D) geometry. These solitons are quantum bound states of a mesoscopic gas, which opens up fascinating possibilities: apart from testing mean field predictions in these systems [3,4], one can address truly quantum problems, issuing from the quantum nature of the gas center of mass.

In particular, it was proposed to use a Bose-Einstein condensate in interferometric experiments to test decoherence mechanisms not predicted by usual quantum mechanics and that would show up for very massive particles [5]. Experiments have succeeded in observing interferences with molecules as big as fullerenes, and there is a need for more massive interferometric objects [6]. A soliton with a small number of 100 ^{7}Li atoms has the same mass as C_{60}, with appealing new features: it does not have internal bound states other than its ground state, it can be reversibly dissociated in an unbound atomic gas via a Feshbach resonance, and it allows the exploration of a new regime, in which the center of mass kinetic energy of the interfering object is of the same order as the binding energy of its constituents.

Thanks to the low temperatures accessible in atomic gases, down to 0.45 nK [7], and the weak decoherence present in these systems [8], one may hope to split the center of mass wave function of the solitonic gas in two wave packets that would keep their mutual coherence over mesoscopic distances, say, a fraction of a millimeter, much larger than the size of the soliton. The gas would then have simultaneously nonzero probability amplitudes of being in two different spatial locations, thus forming a mesoscopic quantum nonlocal state. One may then ascertain the presence of such a state by recombining and interfering the two spatial components of the gas. This would constitute a generalization to many atoms of the one-ion experiment of [9]. While mesoscopic superposition states have been observed for radiation fields [10], they have not been reported yet with ultracold atoms, and atom optics with a quantum soliton are a promising alternative to other ideas for their production in these systems [11].

The dynamics of the center of mass wave packet during the scattering of the soliton on a barrier raises nontrivial theoretical issues, since the presence of the barrier makes the 1D many-body problem nonintegrable via the Bethe ansatz. We thus construct an approximate effective low-energy Hamiltonian for the center of mass of the gas, and we derive a rigorous upper bound on the resulting error.

The starting point is the many-body Hamiltonian in 1D, for N bosonic particles of mass m interacting via the usual contact interaction of coupling constant g, in presence of the barrier potential $U(x) \geq 0$:

$$H = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + U(x_i)\right] + g\sum_{i<j}\delta(x_i - x_j). \qquad (1)$$

This is conveniently rewritten as $H = P^2/(2M) + H_{\rm in} + V$, singling out the kinetic energy of the center of mass ($M = Nm$ is the total mass and P the total momentum of the gas), the so-called *internal* Hamiltonian $H_{\rm in}$ and the sum of the N barrier potentials, V. Without a barrier ($V \equiv 0$) there is full separability between the center of mass and the internal variables, so that we split the Hilbert space as a tensorial product of center of mass and internal variables. $H_{\rm in}$ is diagonalized with the Bethe ansatz [12,13]: its ground state is its single discrete eigenstate, the quantum soliton $|\phi\rangle$ of energy $E_0(N)$ [14] and wave function $\phi \propto \exp(-\sum_{i<j} m|g||x_i - x_j|/2\hbar^2)$, separated from a continuum of solitonic fragments by an energy gap which is minus the chemical potential, $|\mu| = E_0(N-1) - E_0(N) = mg^2N(N-1)/(8\hbar^2)$. In the presence of a barrier, we consider the scattering state $|\Psi\rangle$ of the soliton with an incoming center of mass wave vector $K > 0$. We restrict to a *low* K value to have *elastic* scattering [15],

$$E - E_0 \equiv \frac{\hbar^2 K^2}{2M} < |\mu|. \qquad (2)$$

Far from the barrier, one can then observe only a nonfragmented soliton, to the right with the transmission amplitude t, to the left with the reflection amplitude r.

In this elastic regime, an effective Hermitian Hamiltonian may be defined, introducing the projector $\mathcal{P} = I_{\rm CoM} \otimes |\phi\rangle\langle\phi|$ acting as the identity on the center of mass and projecting the internal state on its ground state:

$$\mathcal{P}|\Psi\rangle = |\Phi\rangle \otimes |\phi\rangle. \qquad (3)$$

Far from the barrier, $\Phi(X)$ is simply the center of mass wave function, X being the center of mass position. The so-called $\mathcal{P}G\mathcal{P}$ formalism, where G is the resolvent of the full Hamiltonian [16], then gives the exact equation

$$\frac{\hbar^2K^2}{2M}|\Phi\rangle = \left[\frac{P^2}{2M} + \bar{V}(X) + \delta V\right]|\Phi\rangle. \qquad (4)$$

The first contribution to the effective potential, in the right-hand side of (4), is the convolution of the barrier potential with the internal density profile of the soliton:

$$\bar{V}(X) = \langle\phi|V|\phi\rangle = \int_{-\infty}^{+\infty} dx U(X-x)\rho(x|0), \qquad (5)$$

where $\rho(x|0)$ is the mean density of particles in the soliton knowing that the center of mass is localized in $X = 0$. It was calculated with the Bethe ansatz [17] and is well approximated for $N \gg 1$ by the mean field density profile $\rho(x|0) \simeq N/[4\xi\cosh^2(x/2\xi)]$, where the mean field soliton size is $\xi = \hbar^2/(m|g|N)$. The second contribution in (4) involves virtual transitions to internal excited states:

$$\delta V = \left\langle\phi\left|V\mathcal{Q}\frac{\mathcal{Q}}{E\mathcal{Q} - \mathcal{Q}H\mathcal{Q}}\mathcal{Q}V\right|\phi\right\rangle, \qquad (6)$$

where $\mathcal{Q} = I - \mathcal{P}$. We shall neglect this contribution but not without a justification. From the fact that $\mathcal{Q}H\mathcal{Q} \geq E_0 + |\mu|$, a consequence of the positivity of $P^2/2M$ and V, and of the energy gap of $H_{\rm in}$, we see in the regime (2) that the operator $-\delta V$ is positive and bounded as

$$-\delta V \leq W(X) \equiv \frac{\langle\phi|V^2|\phi\rangle - \bar{V}(X)^2}{|\mu| - \hbar^2K^2/2M}. \qquad (7)$$

When one neglects δV in (4), the exact $\Phi(X)$ is replaced by $\Phi_0(X)$, which involves the same incoming wave e^{iKX}, but outgoing waves $e^{iK|X|}$ whose transmission and reflection amplitudes t_0 and r_0 are only approximate. We have bounded the resulting errors, here only for an even barrier $U(x) = U(-x)$. Introducing the "small parameter," $\epsilon \equiv M\langle\Phi_0|W(X)|\Phi_0\rangle/(\hbar^2K|t_0|)$, we have for $\epsilon < 1/2$

$$|t - t_0| \quad \text{and} \quad |r - r_0| \leq \frac{|t_0|\epsilon}{1 - 2\epsilon}. \qquad (8)$$

It remains to calculate $W(X)$. This is out of reach of mean field theory. We have derived from the Bethe ansatz the large N asymptotic expression [18]

$$W(X) \simeq \frac{2N\xi^4}{|\mu| - \frac{\hbar^2K^2}{2M}} \int_{-\infty}^{+\infty} dx \int_x^{+\infty} dy U''(X + x\xi) \times U''(X + y\xi)\frac{2 + y - x}{(e^y + 1)(e^{-x} + 1)}. \qquad (9)$$

In practice, the barrier $U(x)$ is produced with a Gaussian laser beam, $U(x) = U_0\exp(-2x^2/b^2)$, with a waist b much larger than the soliton size ξ. Then the mean potential $\bar{V}(X)$ is close to $NU(X)$. We shall also assume that the incoming kinetic energy $\hbar^2K^2/2M$ is about half the gap $|\mu| \simeq \hbar^2/8m\xi^2$, so that (2) is satisfied without paying the price of very slow soliton velocities. Then $Kb \gg 1$ and the scattering is in the semiclassical regime: approximate expressions for t_0 and r_0 [19] predict a transmission probability 1/2 for an incident wave vector K_0 such that

$$\frac{\hbar^2K_0^2}{2M} = \max_X \bar{V}(X) \simeq NU_0. \qquad (10)$$

In the vicinity of $K = K_0$, the transmission probability varies sharply from zero to unity,

$$|t_0|^2 \simeq \frac{1}{1 + \exp[\frac{K_0 - K}{\delta K}]} \quad \text{with} \quad \delta K \simeq \frac{1}{\pi\sqrt{2}b}. \qquad (11)$$

It remains to estimate the bound (8). One may take $U'' \simeq U''(X)$ in (9), since $b \gg \xi$, so that

$$W(X) \simeq \frac{N\xi^4}{|\mu| - \frac{\hbar^2K^2}{2M}}[U''(X)]^2\left[\frac{2\pi^2}{3} + 4\zeta(3)\right]. \qquad (12)$$

In $K = K_0$, for $\epsilon \ll 1$, a semiclassical calculation gives

$$|t - t_0| \lesssim \frac{10(\xi/b)^3}{N^{1/2}}\ln(Nb^2/\xi^2), \qquad (13)$$

a quantity checked to be $\ll 1$ in what follows.

We now study the experimental feasibility. A Gaussian laser beam confines $N \simeq 100$ atoms of ^{7}Li in the y-z plane, with a transverse harmonic oscillator length $a_\perp = (\hbar/m\omega_\perp)^{1/2} \simeq 0.54\ \mu$m, where $\omega_\perp \simeq 2\pi \times 4.8$ kHz is the transverse oscillation frequency. In this optical wave guide, the interacting gas has a 1D character if $2\xi \gg a_\perp$. In order to make cooling of the gas not too challenging, we take a not too large soliton length $\xi \simeq 0.9\ \mu$m; the resulting 3D scattering length, $a \simeq -a_\perp^2/(2N\xi) \simeq -1.72$ nm, is in the interval of values $(-\infty, -1.5$ nm) accessible with the Feshbach resonance [1]. Initially the gas is also harmonically trapped along x with an oscillation frequency ω. To prepare the center of mass in a pure state, as required for our coherent splitting and recombination scheme, the gas has to be cooled to extremely low temperatures, here $T = 0.45$ nK [7]. The axial trap is weak, $\hbar\omega < |\mu|/10$, not to affect the internal solitonic variables, and strong enough that the center of mass, still separable in a harmonic trap, has a negligible probability $\exp(-\hbar\omega/k_BT) < 1/10$ to be in an excited state. These two constraints impose the weak value $\omega \simeq 2\pi \times 23.5$ Hz. They also imply $|\mu|/k_BT \simeq 25$, so that the internal variables of the soliton are frozen in their ground state.

At $t = 0$, the gas is launched with a total momentum $\hbar K_0$ such that

$$\frac{\hbar^2K_0^2}{2M} = \frac{|\mu|}{2} \simeq \frac{\hbar^2}{16m\xi^2}. \qquad (14)$$

The corresponding velocity is $\hbar K_0/M \simeq 0.37$ mm/s. Simultaneously the axial trap is switched off, to free the center of mass of the gas, with an initial wave packet

$$\Phi(X) \propto e^{iK_0X} e^{-(X-X_0)^2(\Delta K)^2}. \tag{15}$$

For a sudden opening of the axial trap [20], the Gaussian factor in (15) is the ground state center of mass wave function in the trap, so that $\hbar^2(\Delta K)^2/2M = \hbar\omega/4$, and the wave packet is quasimonochromatic, $\Delta K/K_0 \simeq 0.22$. Even smaller values of ΔK may be obtained by a clever opening procedure of the trap, within times $\sim 1/\omega$ [21]. The wave packet is scattered on a broad Gaussian barrier centered in $x = 0$ (here $X_0 < 0$), a beam splitter, created by a laser beam of waist $b = 5\xi \gg \xi$ and of intensity adjusted to satisfy the half-transmission condition (10). In any realistic case, ΔK remains much larger than δK, so the wave packet experiences a mere filtering in Fourier space, the components with $K > K_0$ being transmitted and the ones with $K < K_0$ being reflected [22]. As a consequence, the wave packet also splits in real space in a transmitted part and a reflected part, which nicely separate since their mean velocity exceeds their spreading velocity: a mesoscopic nonlocal superposition is born.

How to prove this experimentally? First, one checks the absence of fragmentation: a photo of the gas by absorption imaging should always show that *all* the particles are clustered in a single lump of size ξ, randomly situated to the left or to the right of the beam splitter. Second, one checks that the two wave packets are coherent, by recombining them and looking for interference fringes, with a fringe spacing π/K_0. The recombination of the two wave packets is obtained by their total reflection on *mirrors*, produced by two Gaussian laser beams centered in $x = \pm L/2$, $L \gg 1/\Delta K$, with the same waist as the beam splitter but with a higher intensity (say, twice as high). The reflected wave packets interfere around $x = 0$, the beam splitter being switched off [23].

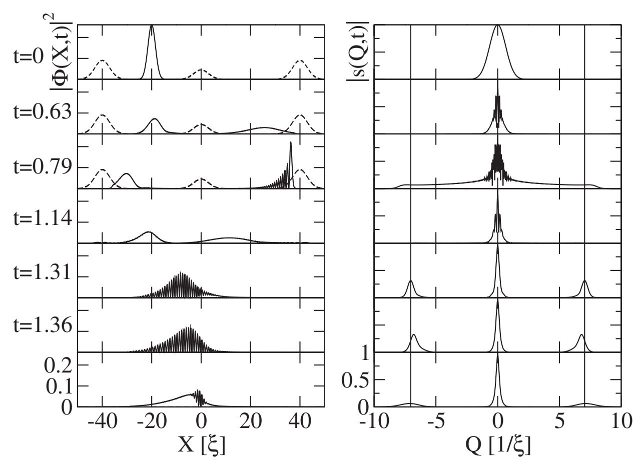

FIG. 1. Evolution of the center of mass wave function of a solitonic gas with $N = 99$ atoms, by integration of (16) for the initial condition (15), with $\Delta K \simeq 0.093K_0$, $X_0 = -15\xi$. Left panel: $|\Phi(X, t)|^2$ (solid lines); effective potential $\bar{V}(X, t)$ (dashed lines). Right panel: modulus of the Fourier transform $s(Q, t)$ of $|\Phi(X, t)|^2$; the vertical lines are in $Q = \pm 2K_0$. The time is in units of $ML/\hbar K_0$ (here $L = 80\xi$). At $t = 0.63$ the state is a nonlocal superposition. At $t = 1.31$ the two reflected spatial components strongly interfere; two narrow peaks of height 0.315 emerge on $|s(Q)|$ in $Q = \pm 2K_0$. Maximal interference occurs at $t = 1.36$. The last line is an average over a Poisson distribution for N, with a mean value $\bar{N} = 99$; the peak height in $|s(Q)|$ is reduced to 0.062.

To study the proposed experiment we have solved Schrödinger's equation for the center of mass wave function, with the initial condition (15) and with the approximation $\delta V = 0$ to the effective Hamiltonian (4) [24]:

$$i\hbar\partial_t\Phi(X, t) = \left[-\frac{\hbar^2}{2M}\partial_X^2 + \bar{V}(X, t)\right]\Phi(X, t). \tag{16}$$

The probability distribution $|\Phi(X, t)|^2$ is plotted at key times in Fig. 1. To quantify the contrast of the interference fringes, we also plotted the modulus of its Fourier transform, $s(Q, t) = \int_{-\infty}^{+\infty} dX e^{-iQX}|\Phi(X, t)|^2$. When the two wave packets overlap, sharp peaks in $|s(Q)|$ indeed form in $Q \simeq \pm 2K_0$, with a contrast $|s(\pm 2K_0)| \simeq 0.32$. This is a high value, as the ideal case of two overlapping plane waves $\Phi(X) \propto e^{iK_0X} + e^{-iK_0X}$ gives 1/2.

The high contrast interference fringes in Fig. 1 are, however, for the center of mass probability distribution, not for the atomic density, which raises the question of their observability by usual fluorescence imaging. The mean atomic density $\rho(x)$ is the convolution of $|\Phi(X)|^2$ with the internal soliton density $\rho(x|0)$; since the soliton size ξ is as large as the fringe spacing π/K_0, one finds that the contrast of the fringes in $\rho(x)$ is several orders of magnitude smaller than in $|\Phi(X)|^2$. This problem can be solved by increasing, just before imaging, the power of the transverse trapping laser by a factor $\simeq 21$, to reduce the transverse harmonic oscillator length to $\tilde{a}_\perp = 0.25$ μm and bring the soliton close to its collapse threshold $N|a|/\tilde{a}_\perp \simeq 0.67$ [25]. Furthermore fluorescence imaging can be optimized to measure directly the quantity $|s(2K_0)|$, by exciting the gas with a laser standing wave along x, produced by the superposition of two laser waves of wave vectors $\vec{k}_\pm = (\pm K_0, k_y, 0)$. The resulting fluorescence rate in direction $\vec{n}$ per unit of solid angle is given in the Born approximation by $d\Gamma/d\Omega \propto \langle|\sum_{i=1}^{N} e^{-ik\vec{n}\cdot\vec{r}_i}e(\vec{r}_i)|^2\rangle$ [26], where $e(\vec{r})$ is the laser electric field. The fluorescence rate Γ_Ω in the solid angle Ω of the detection lens is an oscillating function of the location of the antinodes of the laser standing wave with respect to the interference pattern in $|\Phi(X)|^2$, with a contrast

$$\frac{\Gamma_\Omega^{\max} - \Gamma_\Omega^{\min}}{\Gamma_\Omega^{\max} + \Gamma_\Omega^{\min}} = |s(2K_0)|S_{\text{in}}(\Omega). \tag{17}$$

The reduction factor $S_{\text{in}}(\Omega)$ is a function of the 3D static structure factor of the soliton for fixed center of mass position, which we approximate with the 3D mean field theory. By using a lens of optical axis along $\vec{k}_+ + \vec{k}_-$ with a numerical aperture 0.4, one finds the remarkably high value $S_{\rm in}(\Omega) = 0.84$, thanks to a super-radiant effect [26], which also concentrates 16% of the fluorescence in the 4% solid angle fraction collected by the lens.

It remains to check that decoherence is negligible during the transit time $t_{\rm trans} = ML/\hbar K_0 \lesssim 200$ ms of the nonlocal state in the interferometer. In cold atom experiments, the main source of decoherence is particle losses: A *single* loss event would destroy the quantum superposition, since it "measures" the positions of one or several atoms and localizes the center of mass of the gas within the soliton size ξ. The usual loss rate formula for m-body loss is $dN/dt = -K_m \int d^3 r n^m(\vec{r})$; here one should take for n the 3D density profile for a fixed center of mass position, which we approximate with the mean field theory. For one-body losses due to collisions with the background gas, one should have a loss probability $K_1 N t_{\rm trans} < 1/10$, which imposes the reasonable lifetime $K_1^{-1} > 200$ s. For three-body losses due to formation of deeply bound dimers, the loss constant K_3 for ^{7}Li at the considered magnetic field B is not known. Since $|a|$ is smaller than the Van der Waals length 3 nm, as it is for $B = 0$, we use the $B = 0$ prediction of [27], applying the factor 6 reduction for a condensate, $K_3 \approx 3 \times 10^{-41}$ m^6/s, which leads to a negligible loss event probability $\frac{1}{3}|dN/dt| t_{\rm trans} \approx 0.03$.

In present experiments the number of atoms N fluctuates around the desired mean value $\bar{N}$. Since the launch velocity $\hbar K_0/M$ is fixed, K_0 is proportional to N and also fluctuates [28]. A first side effect is that the half-transmission probability condition may be violated for $N \neq \bar{N}$; fortunately this is not the case for a broad barrier $b \gg \xi$, since both terms of (10) are proportional to N. A second side effect is that the fringe spacing π/K_0 fluctuates, which may blur the fringes. A simple way to estimate this is to assume that $|\Phi(X)|^2 \propto |e^{iK_0X} + e^{-iK_0X}|^2 e^{-X^2/2\sigma^2}$ at the overlap time. Averaging over a Poisson distribution for N with σ and K_0/N fixed leads to, for $|X| \ll \pi\bar{N}/\bar{K}_0$,

$$\langle|\Phi(X)|^2\rangle \simeq \frac{e^{-X^2/2\sigma^2}}{(2\pi)^{1/2}\sigma}[1 + e^{-X^2/2\sigma_c^2}\cos(2\bar{K}_0X)].$$

The fringes persist around the origin over a distance $\sigma_c = \bar{N}^{1/2}/(2\bar{K}_0) = \sqrt{2}\xi$. $|s(2\bar{K}_0)|$ is then reduced by a factor $\sigma_c/(\sigma^2 + \sigma_c^2)^{1/2}$. Estimating σ from Fig. 1 leads to a reduction factor 5 close to the numerical one (see Fig. 1).

In conclusion, we propose to produce a coherently bilocalized gas by scattering an atomic quantum soliton on a barrier. This raises challenging experimental aspects of preparation and detection, but also nontrivial theoretical aspects since this is a many-body problem. We find that a gas with $N \simeq 100$ ^{7}Li atoms can be prepared in a coherent superposition of being at two different locations separated by $\sim$100 μm, and that this can be proved by an interferometric measurement.

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