Partially Integrable Dynamics of Hierarchical Populations of Coupled Oscillators

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We consider oscillator ensembles consisting of subpopulations of identical units, with a general heterogeneous coupling between subpopulations. Using the Watanabe-Strogatz ansatz, we reduce the dynamics of the ensemble to a relatively small number of dynamical variables plus constants of motion. This reduction is independent of the sizes of subpopulations and remains valid in the thermodynamic limits. The theory is applied to the standard Kuramoto model and to the description of two interacting subpopulations, where we report a novel, quasiperiodic chimera state.

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Large populations of coupled oscillators occur in a variety of applications and models of natural phenomena, ranging from collective dynamics of multimode lasers and Josephson junction arrays to the pedestrian synchrony [1]; the analysis of the dynamics of these systems is a topic of high interest. Even in the context of the simplest, paradigmatic case of globally (all-to-all) connected, sine-coupled phase oscillators (the famous Kuramoto model and its generalizations), many problems remain yet unsolved, especially those related to a heterogeneous coupling and nontrivial collective dynamics. In this Letter, we treat an important case of a *hierarchically* organized population. It can be viewed as a (finite or infinite) collection of interacting subpopulations, each consisting of a (finite or infinite) number of identical units; sizes of the subpopulations and couplings between them are generally different (cf. [2,3], and references therein). Using the seminal approach of Watanabe and Strogatz (WS) [4], we demonstrate that each subpopulation can be described by only three dynamical variables plus constants of motion, determined by initial conditions. This partial integrability allows us to separate the full dynamics into a relatively small number of generally dissipative modes (their number is proportional to the number of subpopulations) with possibly nontrivial behavior and the constants of motion. In the thermodynamic limit where the number of subpopulations tends to infinity, we describe the ensemble by integral equations. Remarkably, these equations contain as a particular case the recent ansatz by Ott and Antonsen (OA) [3], which corresponds to the uniform distribution of the constants of motion. With our general formulation we extend the OA ansatz and determine the conditions of its validity. Furthermore, using derived equations, we revise the recent analysis of two interacting subpopulations by Abrams *et al.* [5], where the authors found periodic chimera states using the OA ansatz. Again, with our theory we go beyond this ansatz to the most general case of nonuniformly distributed constants of motion and demonstrate that the dynamics of chimera states is generally quasiperiodic.

Our basic model is a generalization of the Kuramoto model [1] (cf. [2,3]):

$$\frac{d\phi_k^a}{dt} = \omega_a + \frac{1}{N}\sum_{b=1}^{M}\sum_{j=1}^{N_b}\varepsilon_{a,b}\sin(\phi_j^b - \phi_k^a - \alpha_{a,b}). \quad (1)$$

Here we denote the subpopulations by indices $a, b = 1, \ldots, M$. Variable $\phi_k^a(t)$ is the phase of oscillator k in subpopulation a; $k = 1, \ldots, N_a$, where N_a is the size of the subpopulation, and ω_a is the natural frequency of its oscillators (we remind the reader that all oscillators in a subpopulation are *identical*). The total number of oscillators is $N = \sum N_a$, and two constants ε and α describe the coupling with an arbitrary phase shift; cf. [6]. The system can be rewritten as

$$\frac{d\phi_k^a}{dt} = \omega_a + \mathrm{Im}(Z_a e^{-i\phi_k^a}), \quad (2)$$

$$Z_a = \sum_b n_b \varepsilon_{a,b} e^{-i\alpha_{a,b}} r_b e^{i\Theta_b}, \quad (3)$$

where Z_a is the effective force acting on the oscillators of subpopulation a. Here we have introduced the relative population sizes $n_a = N_a/N$ and the complex mean fields for each subpopulation

$$X_a + iY_a = r_a e^{i\Theta_a} = N_a^{-1}\sum_{k=1}^{N_a} e^{i\phi_k^a}. \quad (4)$$

Note that all oscillators in a subpopulation obey the same equation, though generally they have different initial conditions $\phi_k^a(0)$. Thus, we can apply to *each subpopulation* the WS ansatz [4] that reduces the dynamics of the subpopulation to that of three variables $\rho_a(t)$, $\Psi_a(t)$, and $\Phi_a(t)$, via the transformation

$$\tan\left[\frac{\phi_k^a - \Phi_a}{2}\right] = \frac{1-\rho_a}{1+\rho_a}\tan\left[\frac{\psi_k^a - \Psi_a}{2}\right] \quad (5)$$

containing N_a constants ψ_k^a, which are directly determined from the initial state $\phi_k^a(0)$ and additionally satisfy

$$\sum_{k=1}^{N_a}\cos\psi_k^a = \sum_{k=1}^{N_a}\sin\psi_k^a = 0. \quad (6)$$

Because of an arbitrary shift of constants ψ_k with respect to Ψ, only $N_a - 3$ of constants ψ_k^a are independent. The WS method is valid generally, provided the number of oscillators in a subpopulation is larger than 3, and the initial state does not have too large clusters; see [4] for a detailed discussion of these conditions and of how $\rho_a(0)$, $\Psi_a(0)$, $\Phi_a(0)$, and ψ_k^a can be computed from $\phi_k^a(0)$. With account of Eq. (3), we write the WS equations for our setup as

$$\frac{d\rho_a}{dt} = \frac{1-\rho_a^2}{2}\,\mathrm{Re}(Z_a e^{-i\Phi_a}), \qquad (7)$$

$$\frac{d\Psi_a}{dt} = \frac{1-\rho_a^2}{2\rho_a}\,\mathrm{Im}(Z_a e^{-i\Phi_a}), \qquad (8)$$

$$\frac{d\Phi_a}{dt} = \omega_a + \frac{1+\rho_a^2}{2\rho_a}\,\mathrm{Im}(Z_a e^{-i\Phi_a}). \qquad (9)$$

In order to illustrate the physical meaning of the new variables, let us consider how they characterize the distribution of the phases of a subpopulation. Generally, oscillators form a bunch, and the amplitude ρ characterizes its width: $\rho = 0$, if the distribution is uniform (asynchrony), and $\rho = 1$, if the distribution shrinks to the δ function (full synchrony). Amplitude ρ is roughly proportional to the amplitude of the mean field r [see Eq. (4)] in the sense that $\rho = r = 0$ for the full asynchrony and $\rho = r = 1$ for the full synchrony. For intermediate cases, these quantities generally differ and coincide only in a special case, outlined below. The phase variable Φ characterizes the position of the bunch and is therefore related to the phase of the mean field: $\Phi \approx \Theta$. Another phase variable Ψ describes the shift of individual oscillators with respect to the bunch (see Fig. 3 in [4]; generally, the oscillators can move with a velocity different from that of the bunch [7]).

The set of Eqs. (7)–(9) is a straightforward generalization of the WS equations [4] to the case of M interacting subpopulations. For a further analysis, and, in particular, for the consideration of the thermodynamic limit, it is convenient to introduce new variables, a phase shift $\zeta_a = \Phi_a - \Psi_a$, and a complex bunch amplitude $z_a = \rho_a e^{i\Phi_a}$. Then we can rewrite Eqs. (7)–(9) as

$$\frac{dz_a}{dt} = i\omega_a z_a + \frac{1}{2}Z_a - \frac{z_a^2}{2}Z_a^*, \qquad (10)$$

$$\frac{d\zeta_a}{dt} = \omega_a + \mathrm{Im}(z_a^* Z_a). \qquad (11)$$

Next, we have to represent the complex force Z_a [see Eq. (3)] in terms of new variables. For this goal it is convenient to rewrite Eq. (5) in an equivalent form $e^{i\phi_k} = e^{i\Phi}(\rho e^{i\Psi} + e^{i\psi_k})/(\rho e^{i\psi_k} + e^{i\Psi})$. Substituting this into Eq. (4), we obtain

$$r_a e^{i\Theta_a} = \rho_a e^{i\Phi_a}\gamma_a(z_a, \zeta_a) = z_a\gamma_a(z_a, \zeta_a),$$
$$\gamma_a(z_a, \zeta_a) = \frac{1}{N_a}\sum_{k=1}^{N_a}\frac{1 + |z_a|^{-2} z_a^* e^{i(\zeta_a + \psi_k^a)}}{1 + z_a^* e^{i(\zeta_a + \psi_k^a)}}. \qquad (12)$$

From Eq. (10) it follows that the dynamics of the complex bunch amplitude of a subpopulation z_a is determined by the force Z_a, resulting from interaction within the subpopulation as well as from interaction with other subpopulations. Contributions to Z_a are proportional to the relative weights n_b, to the coupling constant $\varepsilon e^{-i\alpha}$, and to the complex mean field $\gamma_b(z_b, \zeta_b)z_b$, which generally depends not only on the global variables ζ_b and z_b but also on the constants of motion ψ_k^b. Equations (10) and (11), as well as equivalent Eqs. (7)–(9), together with the definitions (3) and (12) are exact and complete; they show that the dynamics of a hierarchical ensemble of oscillators can be reduced to $3M$ ordinary differential equations (ODEs) plus $N - 3M$ constants of motion.

In most applications of the theory, one treats infinite ensembles [1], and thus we discuss how a thermodynamic limit $N \to \infty$ can be introduced in this picture. There are two main ways of performing this.

(i) Suppose that the number of subpopulations M remains finite, but their sizes grow $N, N_a \to \infty$ in a way that $n_a = \mathrm{const}$. In this case only Eq. (12) is affected and should be now written as an integral

$$\gamma_a(z_a, \zeta_a) = \int_{-\pi}^{\pi}\frac{1 + |z_a|^{-2} z_a^* e^{i(\zeta_a + \psi)}}{1 + z_a^* e^{i(\zeta_a + \psi)}}\sigma_a(\psi)d\psi. \qquad (13)$$

Here $\sigma_a(\psi)$ is the distribution of the constants of motion ψ in the subpopulation a, and additionally it satisfies [cf. (6)]

$$\int_{-\pi}^{\pi}\sigma_a(\psi)e^{i\psi}d\psi = 0. \qquad (14)$$

In this limit the ensemble is described by a set of $3M$ ODEs, where the right-hand sides depend on the variables via integrals (13). The integrals of motion are now the functions $\sigma_a(\psi)$.

(ii) In another limiting case, we keep the size of each subpopulation N_a finite but let the number of subpopulations grow $M \to \infty$. Considering indices a and b as continuous variables, we write instead of Eq. (3)

$$Z(a) = \int dbn(b)\varepsilon(a,b)e^{-i\alpha(a,b)}\gamma(b)z(b). \qquad (15)$$

Now Eqs. (10)–(12) and (15) become a system of integral equations; still it is simpler than the original Eq. (1) as at each value of the continuous parameter a we have only three real time-dependent variables.

Certainly, one can also perform both thermodynamic limits simultaneously. Then the ensemble is described by the system (10), (11), (13), and (15).

Remarkably, Eq. (10) coincides with the basic equation of OA theory [3]; however, there it appears without Eq. (11). To clarify this issue, we study an important case when Eqs. (10) and (11) *decouple*. To this end we represent the fraction in Eqs. (12) and (13) as a series

$$\gamma_a = 1 + (1 - |z_a|^{-2})\sum_{l=2}^{\infty} C_l^a(-z_a^* e^{i\zeta_a})^l, \qquad (16)$$

where complex constants C_l^a depend only on the distribution of the constants of motion

$$C_l^a = \frac{1}{N_a}\sum_{k=1}^{N_a} e^{il\psi_k^a} \quad \text{or} \quad C_l^a = \int_{-\pi}^{\pi} \sigma_a(\psi) e^{il\psi} d\psi, \quad (17)$$

and we used that $C_1^a = 0$ due to Eqs. (6) and (14). Obviously, the governing equations simplify, if $C_l^a = 0$ for $l \geq 2$ and all a, and, hence, $\gamma = 1$. Then the force Z does not depend on the phase variable ζ, and Eq. (10) decouples from Eq. (11). It is easy to see from Eqs. (17) that C_l^a, which are in fact Fourier coefficients of the distribution of the constants of motion ψ, vanish in the thermodynamic limit of type (i), if $\sigma(\psi) = 1/2\pi$. However, if the number of oscillators in a subpopulation N_a is finite, then, even for a uniform spreading of ψ_k, the discrete sum in (17) yields $|C_l^a| = 1$, $\arg(C_l^a) = \psi_1^a$, for $l = N_a, 2N_a, \ldots$, and we get

$$\gamma_a = 1 + \frac{1 - |z_a|^2}{1 - [-z_a^* e^{i(\zeta_a + \psi_1^a)}]^{N_a}} [-z_a^* e^{i(\zeta_a + \psi_1^a)}]^{N_a}. \quad (18)$$

Thus, the deviation of γ_a from unity is exponentially small in the size of the subpopulation and, therefore, can be neglected for large N_a. This is exactly the case where the complex bunch amplitude ρ is equal to the mean field amplitude r, because in (12) $\gamma = 1$.

Hence, for the uniform distribution of constants of motion ψ, ensemble (1) admits a simplified description via Eq. (10), supplemented by an equation for Z_a, either in a discrete or in a continuous [for the thermodynamic limit of type (ii)] form:

$$Z_a = \sum_b n_b \varepsilon_{a,b} e^{-i\alpha_{a,b}} z_b, \quad (19)$$

$$Z(a) = \int db n(b) \varepsilon(a,b) e^{-i\alpha(a,b)} z(b). \quad (20)$$

A relation between the distribution of the original phases ϕ_k and the uniform distribution of constants of motion ψ_k follows from Eq. (5): One can see that different distributions of the phases ϕ_k, parametrized by different values of ρ, correspond to the uniformly distributed constants ψ_k.

As a first application of our framework, we apply Eqs. (10)–(12) and (15) to the classical Kuramoto problem (cf. [3]). We set $\varepsilon(a,b) = \varepsilon = \text{const}$, $\alpha = 0$, use the frequency as the subpopulation index, $a = \omega$, and perform the thermodynamic limit (ii). As a result, in the case when $\gamma = 1$ and the variable ζ (as well as the constants of motion) do not influence the dynamics, we obtain exactly Eqs. (10) and (20), derived recently by Ott and Antonsen [3] under an assumption of a certain parametrization of the phase distribution. Considering the Lorentzian distribution of natural frequencies $n(\omega) = [\pi(\omega^2 + 1)]^{-1}$ and using analytic properties of $z(\omega)$ as a function of complex frequency ω, OA have calculated the integral in Eq. (20) by the residue of the pole at $\omega = i$ and have obtained $Z = \varepsilon z(i)$. Substituting Z into Eq. (10) for $\omega = i$, OA derived a closed equation for $\tilde{Z} = Z/\varepsilon$, i.e., for the usual Kuramoto mean field of the whole population

$$\dot{\tilde{Z}} = \left(-1 + \frac{\varepsilon}{2}\right)\tilde{Z} - \frac{\varepsilon}{2}|\tilde{Z}|^2\tilde{Z}, \quad (21)$$

solved it, and in this way obtained explicitly the evolution of the mean field.

From our derivation of the equations of motion, we conclude that the particular ansatz used in Ref. [3] corresponds to the case of uniformly distributed constants of motion ψ_k, which is equivalent to vanishing Fourier coefficients C_l. Next we discuss what changes if the distribution of constants ψ_k is not uniform, i.e., $C_l \neq 0$. Let us treat the effect of nonvanishing coefficients C_l perturbatively, assuming that in the first approximation the OA ansatz is valid. Considering for simplicity the effect of $C_2 \neq 0$ only, we obtain a correction to the mean field by substituting (16) into (15):

$$\Delta Z \approx \varepsilon \int d\omega \frac{z^*(\omega)[|z(\omega)|^2 - 1]e^{i2\zeta(\omega)}C_2(\omega)}{\pi(\omega^2 + 1)}. \quad (22)$$

Calculation of this integral by the residue yields

$$\Delta Z \approx \varepsilon z^*(i)[|z(i)|^2 - 1]e^{i2\zeta(i)}C_2(i). \quad (23)$$

From Eq. (11) it follows that in the first approximation $\zeta(i) = \zeta_0 + it$. Therefore $\Delta Z \propto e^{-2t}$. We conclude that the contribution of a nonuniform distribution of constants ψ_k results in an exponentially decaying correction to the mean field. The characteristic time scale of this decay is 1/2, to be compared with the characteristic time scale of the evolution of the mean field, which, according to Eq. (21), is $(\varepsilon/2 - 1)^{-1}$. Thus, close to criticality $\varepsilon_c = 2$, the approximation of vanishing constants C_l works well after short transients; this is not surprising as near a bifurcation point the dynamics is typically effectively low-dimensional, dominated by a few normal modes. Far from criticality the time scale separation is not valid and the dynamics is generally high-dimensional.

As a second example, we extend recent results of Abrams *et al.* [5]. They studied two coupled subpopulations of identical oscillators, i.e., model (1) with $a = 1, 2$, $\omega_1 = \omega_2$, $N_1 = N_2$, and heterogeneous coupling $\varepsilon_{1,1} = \varepsilon_{2,1} = 2\mu$, $\varepsilon_{1,2} = \varepsilon_{2,1} = 2\nu$, and $\alpha_{a,b} = \alpha$, where $\nu = 1 - \mu$. Using the OA ansatz [3], Abrams *et al.* derived equations for the complex order parameters $z_{1,2}$ and analyzed the so-called chimera state, where, e.g., the first subpopulation is fully synchronized ($\rho_1 = 1$), whereas the other one is only partially synchronized ($\rho_2 < 1$); they have found both static and time-periodic solutions for ρ_2. With our approach we describe the system *exactly*, by writing six Eqs. (7)–(9) for both subpopulations. Since we are interested in the chimera state in the second subpopulation, the first, synchronous one is described by its phase Φ_1 only. In this case $Z_1 = \mu e^{i(\Phi_1 - \alpha)} + \nu A e^{i(\Phi_2 + \beta - \alpha)}$ and $Z_2 = \nu e^{i(\Phi_1 - \alpha)} + \mu A e^{i(\Phi_2 + \beta - \alpha)}$, where

$$A(\rho_2, \Psi_2) e^{i\beta(\rho_2, \Psi_2)} = \frac{1}{N_2}\sum_{k=1}^{N_2} \frac{\rho_2 e^{i\Psi_2} + e^{i\psi_k^{(2)}}}{e^{i\Psi_2} + \rho_2 e^{i\psi_k^{(2)}}}. \quad (24)$$

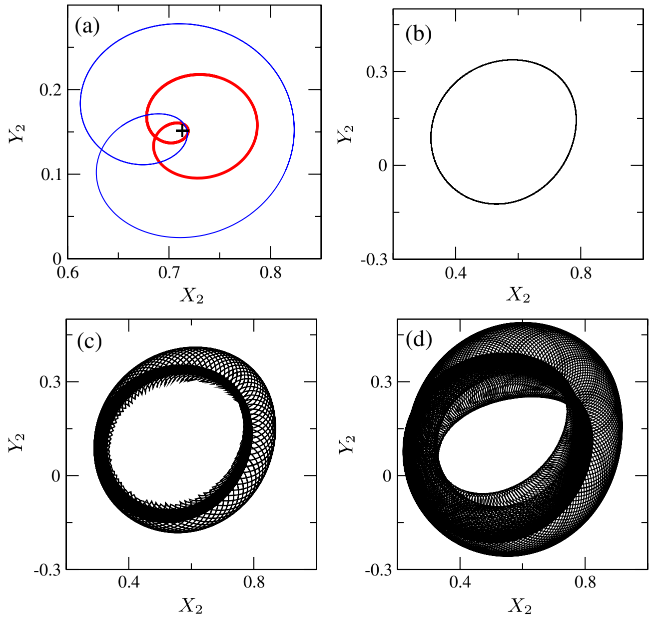

FIG. 1 (color online). Simulation of ensemble (1) for $N_1 = N_2 = 64$, $\alpha = \pi/2 - 0.1$, and different distributions of constants of motion $\psi_k^{(2)}$. Mean fields X_2 and Y_2 are defined by Eq. (4). (a) $\mu = 0.6$; uniform distribution of $\psi_k^{(2)}$ results in the steady state (black plus) (cf. [5]), whereas nonuniform distributions with $q = 0.9$ and $q = 0.7$ yield limit cycle solutions (bold red and blue solid lines, respectively). (b)–(d) $\mu = 0.65$; uniform distribution of $\psi_k^{(2)}$ yields a limit cycle solution (b) (cf. [5]), whereas for (c) $q = 0.9$ and for (d) $q = 0.7$ we observe a new type of the chimera state with a quasiperiodic dynamics.

Next, we note that the dynamics depends only on the phase difference $\delta = \Phi_1 - \Phi_2$ and, hence, write a closed system of three equations:

$$\frac{d\rho_2}{dt} = \frac{1-\rho_2^2}{2}[\mu A\cos(\beta-\alpha) + \nu\cos(\delta-\alpha)], \quad (25)$$

$$\frac{d\delta}{dt} = -\mu\left(\sin\alpha + \frac{1+\rho_2^2}{2\rho_2}A\sin(\beta-\alpha)\right) + \nu\left(A\sin(\beta-\alpha-\delta) - \frac{1+\rho_2^2}{2\rho_2}\sin(\delta-\alpha)\right), \quad (26)$$

$$\frac{d\Psi_2}{dt} = \frac{1-\rho_2^2}{2\rho_2}[\mu A\sin(\beta-\alpha) + \nu\sin(\delta-\alpha)]. \quad (27)$$

Following Abrams *et al.* [5], we take a thermodynamic limit $N_2 \to \infty$. Next, we take a uniform distribution of the constants $\psi^{(2)}$ [which enter only via the relations (24)]—we remind the reader that this choice corresponds to the restriction imposed by OA on the phase distribution in their ansatz. Then from Eq. (24) it follows that $A = \rho_2 = r_2$, $\beta = 0$, and equations for ρ_2 and δ decouple from Eq. (27). The obtained Eqs. (25) and (26) constitute exactly the system analyzed in [5]. For a nonuniform distribution of $\psi_k^{(2)}$, we have to analyze the full three-dimensional system (25)–(27), which certainly can exhibit more complex solutions.

To verify our theoretical prediction, we have performed numerical simulations of the ensemble (1) for the same parameters, where Abrams *et al.* obtained stationary and time-periodic solutions, but for different distributions of the constants ψ_k. Namely, we took ψ_k, uniformly distributed in the range $-q\pi < \psi_k^{(2)} < q\pi$, where $q \le 1$ is a parameter. For $q = 1$ we have reproduced the results of Ref. [5], while for $q < 1$ the dynamics attains an additional time dependence and becomes periodic and quasiperiodic, respectively (see Fig. 1).

In conclusion, we have performed the exact reduction of the dynamics of hierarchically organized populations of coupled oscillators. Because of the partial integrability, only three dynamical variables remain relevant for each subpopulation, and all others are constants of motion. We have demonstrated the power of our formalism by considering two different thermodynamic limits. The first case of infinitely many subpopulations covers, in particular, the Kuramoto problem. Here we have demonstrated that for a particular case of uniformly distributed constants of motion the governing equations decouple and reduce to the recently found particular ansatz of Ott and Antonsen [3]. Considering another thermodynamic limit of two infinitely large subpopulations, we applied our framework to the model by Abrams *et al.* [5] and revealed the existence of novel, quasiperiodically breathing chimera states. Furthermore, the method can be in a straightforward way extended to the cases of nonlinearly coupled populations [7], externally forced ensembles, etc. In these cases even a chaotic dynamics of the global variables can be expected. The main limitation of the theory is that the coupling in Eq. (1) has a sine form.

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