Chiral Logarithms in the Massless Limit Tamed

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We derive nonlinear recursion relations for the leading chiral logarithms (LLs) in massless theories. These relations not only provide a very efficient method of computation of LLs (e.g., the 33-loop contribution is calculated in a dozen of seconds on a PC) but also equip us with a powerful tool for the summation of the LLs. Our method is not limited to chiral perturbation theory only; it is pertinent to any nonrenormalizable effective field theory such as, for instance, the theory of critical phenomena, low-energy quantum gravity, etc.

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The very fact of the spontaneous breakdown of approximate chiral symmetry in strong interactions leads to the possibility of the systematic expansion of hadronic amplitudes and correlation functions at low energies. The expansion is organized in powers of external momenta and the masses of pseudo Nambu-Goldstone bosons (pions). We denote the corresponding expansion parameter generically as p^2. An efficient method to perform the chiral expansion is based on the technique of an effective chiral Lagrangian [1]. The leading $O(p^2)$ hadronic amplitudes can be obtained from the tree diagrams of the famous Weinberg Lagrangian [2]:

$$\mathcal{L}_2 = \frac{F^2}{4}\mathrm{tr}[(\partial_\mu U \partial_\mu U^\dagger) + m^2(U + U^\dagger)], \qquad (1)$$

where F is the pion decay constant in the chiral limit, and m is the pion mass. The chiral corrections of the order $O(p^4 \ln p^2)$ can be obtained from the one-loop calculation with the Weinberg Lagrangian (1) [1,3]. In order to compute higher corrections, e.g. $O(p^4)$, $O(p^6 \ln p^2)$, etc., one has to include terms with four and higher derivatives in the effective chiral Lagrangian [4]. Note, however, that the leading chiral logarithms (LLs), i.e., the correction of the form $O(p^2[p^2 \ln p^2]^n)$ can be obtained from the n-loop diagrams generated by the Weinberg Lagrangian (1). The conciseness and beauty of the leading logarithm approximation lies in the fact that LL corrections depend only on one basic low-energy constant—F. In this approximation one avoids rapid proliferation of the low-energy constant with increasing of the chiral order. The calculation of LLs is a Herculean task—it requires the computation of n-loop diagrams in the *nonrenormalizable* field theory (1). Presently, the LLs are computed to the two-loop accuracy for the $\pi\pi$-scattering amplitude [5], to the five-loop accuracy for the correlator of scalar currents [6], and to the three-loop accuracy for the generalized parton distributions (GPDs) [7]. We note that for the case of the chiral corrections to GPDs the summation of LLs is indispensable [7,8], because the smallness of the chiral expansion parameter is compensated by $1/x_{\mathrm{Bj}}^n$.

In this Letter we present an efficient method to compute LLs in the massless case. This method reduces the task of LLs computation to a simple algebraic problem, and paves a way towards the summation of LLs. We stress that our method only works for observables in the chiral limit, i.e., when the masses in the effective theory vanish. Its generalization for the massive case will be presented elsewhere.

We discuss details of the method for the massless $O(N+1)$ σ-model defined by the Lagrangian

$$\mathcal{L}_2 = \tfrac{1}{2}[\partial_\mu \sigma \partial_\mu \sigma + \partial_\mu \pi^a \partial_\mu \pi^a], \qquad (2)$$

where the fields are constrained by the relation $\sigma^2 + \sum_{a=1}^N \pi^a \pi^a = F^2$. We consider the $O(N+1)$ σ-model for the following reasons: (i) It is equivalent to the massless two-flavour Weinberg Lagrangian (1) for $N = 3$. (ii) It is a free field theory for $N = 1$, which can be used as a check of our calculations. (iii) It can be solved in the large-N limit, which provides a check of our calculations and allows us to asses the accuracy of the $1/N$ expansion without tedious calculations.

For simplicity, we consider LLs for the forward $\pi\pi$ scattering amplitude in the massless $O(N+1)$ σ-model (2). The reader can easily apply our method to an observable in a field theory of her or his choice. The forward amplitude computed at Mandelstam $t = 0$ has the form

$$T^{abcd}(s) = \delta^{ab}\delta^{cd}A(s) + \delta^{cb}\delta^{da}B(s) + \delta^{bd}\delta^{ac}C(s). \qquad (3)$$

The chiral expansion of the functions $A(s)$, $B(s)$, $C(s)$ has the following structure

$$A(s) = (4\pi)^2 S \sum_{n=0}^{\infty}\sum_{k=0}^{n} A_n^{(k)} S^n L^k,$$
$$B(s) = (4\pi)^2 S \sum_{n=1}^{\infty}\sum_{k=0}^{n} B_n^{(k)} S^n L^k, \qquad C(s) = A(-s), \qquad (4)$$

where we introduce a dimensionless invariant energy $S \equiv s/(4\pi F)^2$ and L denotes the chiral logarithms, $L \equiv \ln(\mu^2/s)$, μ is the renormalization scale. The first $A_0^{(0)}$-term in the expression for $A(s)$ corresponds to the tree contribution to the scattering amplitude in the $O(N+1)$ σ-model (2). The other terms are higher chiral order corrections. Our aim consists in the calculation of the LL coefficients $A_n^{(n)} \equiv A_n$ and $B_n^{(n)} \equiv B_n$ appeared in Eq. (4).

Simple power counting [1] shows that the coefficient $A_n^{(k)}$ originates from the k-loop diagram with vertices from the chiral Lagrangian $\mathcal{L}_p$ with the number of derivatives of the pion fields $p \le 2(n+1-k)$. We see that the n-th order LL coefficient receives contribution from the n-loop diagrams with vertices generated by the leading Lagrangian (2). The UV divergencies in a n-loop diagram are removed by the subtraction of lower-loop graphs with insertion of the local counterterms corresponding to the subdivergencies of the original n-loop diagram. See detailed discussion of the structure of the subtractions in Refs. [8,9]. The local counterterms relevant for our calculations renormalize the couplings the all-order Lagrangian, which encodes the structure of counterterms:

$$\mathcal{L} = -\frac{1}{8}\sum_{n=1}^{\infty}\sum_{\substack{j=0\\ \text{even}}}^{n}\frac{g_{nj}(\mu)}{(4\pi F)^{2n}}\partial^{2n}P_j\left(\frac{\partial_1\partial_2}{\partial^2}\right). \tag{5}$$

Here P_j are Legendre polynomials and a convenient notation for the operator monomials is introduced:

$$\partial^{2n}\left(\frac{\partial_1\partial_2}{\partial^2}\right)^j \equiv (\pi^a\overleftrightarrow{\partial}_{\nu_1}\ldots\overleftrightarrow{\partial}_{\nu_j}\pi^a)\partial^{2(n-j)}(\pi^b\overleftrightarrow{\partial}_{\nu_1}\ldots\overleftrightarrow{\partial}_{\nu_j}\pi^b). \tag{6}$$

The coupling constants $g_{nj}(\mu)$ are enumerated by two indices. The index n indicates the number of derivatives of the pion fields (equal to $2n$) in the corresponding counterterm. We refer to the index n as "principal index." The second index j corresponds to the "exchanged spin" of the counterterm. The tree level contribution of the vertices (5) to the amplitude can be easily computed with the result

$$\begin{aligned} A_{\text{tree}}(s) &= -\sum_{n=1}^{\infty}(-S)^n\sum_{\substack{j=0\\ \text{even}}}^{n}g_{nj}(\mu), \\ B_{\text{tree}}(s) &= -\sum_{\substack{n=2\\ \text{even}}}^{\infty}(S)^n g_{nn}(\mu)\frac{(2n)!}{n!n!}. \end{aligned} \tag{7}$$

The expansion coefficients $A_n^{(k)}$ and $B_n^{(k)}$ of the amplitude (4) are functions of the infinite set of couplings $g_{nj}(\mu)$, denoted by $\mathbf{g}$. These coefficients depend on the renormalization scale μ through μ dependence of the couplings $\mathbf{g}$. The renormalized (physical) amplitude is given by the sum of Eq. (4) and (7) and it must be independent of μ. Thus, imposing the requirement that $\mu^2\frac{d}{d\mu^2}(A(s)+A_{\text{ct}}(s)) = 0$ we obtain [10] the following set of equations:

$$\begin{aligned} &A_n^{(1)}(\mathbf{g}) + (-1)^n\sum_{\substack{j=0\\ \text{even}}}^{n+1}\beta_{n+1j}(\mathbf{g}) = 0 \\ &\hat{H}A_n^{(k)}(\mathbf{g}) + (k+1)A_n^{(k+1)}(\mathbf{g}) = 0. \end{aligned} \tag{8}$$

Here the β-functions are defined as $\beta_{nj}(\mathbf{g}) \equiv \mu^2\frac{d}{d\mu^2}g_{nj}(\mu)$. Also we have introduced the differential operator $\hat{H}$ acting on the space of the coupling constants:

$$\hat{H} \equiv \sum_{n=1}^{\infty}\sum_{\substack{j=0\\ \text{even}}}^{n}\beta_{nj}(\mathbf{g})\frac{\partial}{\partial g_{nj}}. \tag{9}$$

The set of Eqs. (8) has the following solution

$$A_n^{(k)}(\mathbf{g}) = \frac{(-1)^{n+k}}{k!}\hat{H}^k\sum_{\substack{j=0\\ \text{even}}}^{n+1}g_{n+1j}. \tag{10}$$

The lowest constant $A_0^{(0)} = g_{10} = 1$ is fixed by the tree level calculations with the Lagrangian (2). We see from the solution (10) that, in order to obtain the LLs (constants $A_n^{(n)}$), we have to apply the operator $\hat{H}$ n times to a linear combination of the coupling constants g_{nj}. This at first glance formidable problems can be solved if one notes the following crucial property of the operator $\hat{H}$:

$$\hat{H}^n g_{mj} = 0, \quad \text{if } m < n. \tag{11}$$

Indeed, the loop diagrams contributing to the renormalization of the constant g_{mj} include vertices with constants g_{pl} with $p < m$ only. It implies that the $\beta_{mj}(\mathbf{g})$-functions depend only on the subset of the low-energy constants g_{pl} with $p < m$. Hence, it is easy to see that the application of the operator $\hat{H}$ to g_{mj} leads to the lowering of the principal index m by one unit; i.e., we can write the general form of the action of $\hat{H}^n$ on g_{n+1j}:

$$\hat{H}^n g_{n+1j} = n!\omega_{nj}g_{10}^{n+1} = n!\omega_{nj}. \tag{12}$$

The expression (12) simultaneously presents the definition of quantities ω_{nj} (n is the number of loops and $j \le (n + \text{Mod}(n,2))$), which determine the LL coefficients $A_n \equiv A_n^{(n)}$ [see Eq. (10) with $k = n$].

Now our aim is to calculate ω_{nj} from Eq. (12). Because of the property (11) only one-loop piece of β_{nj}-functions, which is quadratic in couplings g_{nj}, contribute to Eq. (12). This observation is in accordance with general RG analysis of Ref. [9]. The general structure of the one-loop β_{nj}-function is the following:

$$\beta_{nj}^{1-\text{loop}}(\mathbf{g}) = \sum_{m=1}^{n-1}\sum_{\substack{i=0\\ \text{even}}}^{m}\sum_{\substack{l=0\\ \text{even}}}^{n-m}\mathcal{B}_j^{(m,i)(n-m,l)}g_{mi}g_{(n-m)l}. \tag{13}$$

Here $\mathcal{B}_j^{(m,i)(n-m,l)}$ are numerical coefficients which can be obtained from the calculation of one-loop diagrams shown in Fig. 1. The result of calculations in the dimensional

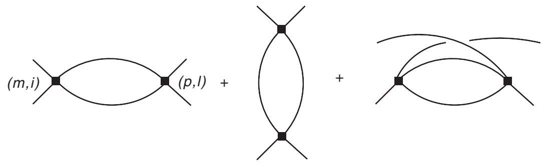

FIG. 1. One-loop diagrams contributing to the β-function's coefficients (14). Filled squares denote the counterterms (mi) and (pl) introduced in Eq. (5).

regularizations and minimal subtraction scheme gives the following result:

$$\mathcal{B}_j^{(m,i)(p,l)} = \frac{1}{2j+1}\left[\frac{N}{2}\delta_{ij}\delta_{lj} + \delta_{ij}\Omega_p^{li} + \delta_{lj}\Omega_m^{il}\right] + (1+(-1)^j)\sum_{k=0}^{\min[p,m]}\frac{\Omega_m^{ik}\Omega_p^{lk}\Omega_{m+p}^{kj}}{2k+1}, \qquad (14)$$

where constants Ω_n^{AB} are computed as the following integral with Legendre polynomials:

$$\Omega_n^{AB} = \frac{2B+1}{2^{n+1}}\int_{-1}^{1} dx P_A\left(\frac{x+3}{x-1}\right)P_B(x)(x-1)^n. \qquad (15)$$

The $(n+1)\times(n+1)$ matrix $\hat{\Omega}_n$ with matrix elements given by Eq. (15), posses many beautiful and intriguing properties. For example, $\hat{\Omega}_n^2 = 1$ and $\mathrm{tr}(\hat{\Omega}_n) = \sin(\pi n)$, which follow from the fact that the matrix $\hat{\Omega}$ represents $SL(2,R)$ transformations. The relation of this symmetry group to the general structure of the renormalization procedure in a wide class of effective quantum field theories will be discussed elsewhere.

Now substituting the one-loop β-function (13) into Eq. (12) we obtain the following nonlinear recursive relations for desired coefficients ω_{nj}:

$$\omega_{nj} = \frac{1}{n}\sum_{m=0}^{n-1}\sum_{\substack{i=0\\ \text{even}}}^{m+1}\sum_{\substack{l=0\\ \text{even}}}^{n-m}\mathcal{B}_j^{(m+1,i)(n-m,l)}\omega_{mi}\omega_{(n-m-1)l}. \qquad (16)$$

The recursion (16) allows us to express the higher coefficients ω_{nj} through that with lower principal indices, starting with $\omega_{00} = 1$. [We remind that n enumerates the loop order and $j \le (n + \mathrm{Mod}(n,2))$] The coefficients of the β-functions $\mathcal{B}_j^{(m+1,i)(n-m,l)}$ are given by Eq. (14). The LLs for the amplitudes $A(s)$, $B(s)$ [$(C_n = (-1)^{n+1}A_n)$] can be computed in terms of ω_{nj} as follows:

$$A_n = \sum_{\substack{j=0\\ \text{even}}}^{n+1}\omega_{nj}, \qquad B_n = \frac{(2n+2)!}{[(n+1)!]^2}\omega_{nn+1}. \qquad (17)$$

Although we demonstrated our method for the scattering amplitude at zero momentum transfer it is easily generalized to the case with nonzero t. The result for the LLs for the scattering amplitude $A(s,t,u)$ [see Eq. (3)] is the following [$t = -\frac{s}{2}(1-\cos\theta)$, $s+t+u=0$]:

$$A(s,t,u) = (4\pi)^2 S\sum_{n=0}^{\infty}\sum_{\substack{j=0\\ \text{even}}}^{n}\omega_{nj}(SL)^n P_j(\cos\theta),$$

where ω_{nj} satisfy the recursive relations (16), amplitudes B and C are obtained by the crossing relation $B(s,t,u) = A(t,s,u)$, $C(s,t,u) = A(u,t,s)$.

The recursive relation (16) is the main result of the present paper. It allows a very fast computation of LLs. For example, the 33-loop chiral LL is computed in a dozen of seconds on a PC [11]. The 6-loop results for LLs are presented [12] in Table I for the amplitude $A(s)$ and 7-loop results in Table II for the amplitude $B(s)$.

Our results for LLs for $N = 3$ agree with two-loop calculations of $\pi\pi$ amplitude [5] and with five-loop results for the correlator of the scalar currents [6]. Additional check of our method is provided by the case of $N = 1$. Indeed, for that value of N the Lagrangian (2) corresponds to a free field theory; therefore, we should obtain nullification of LLs. For $N = 1$ the scattering amplitude is given by the sum $A + B + C$, it is easy to see that all LLs are cancelled in this case.

One can apply our method to the case of renormalizable field theory in which the LLs are summed up by the 1-loop RG equations. In a renormalizable theory the 1-loop β-function (13) involves the charges g_{nj} with the same principal index n; therefore, in our recursive relations (16) only coefficients $\mathcal{B}_j^{(k,i)(k,l)}$ [the principal index k corresponds to the chiral power of the vertex in renormalizable Lagrangian] are present. For example, the $O(N+1)$ symmetric ϕ^4-theory corresponds to $k = 0$ or equivalently $p = m = 0$ and $i = j = l = 0$ in Eq. (14). Simple calcu-

TABLE I. LL coefficients for the amplitude $A(s)$.

# loops	$N = 3$	Arbitrary N
1	$\frac{2}{3}$	$\frac{N}{2}(1 - \frac{5}{3N})$
2	$\frac{25}{18}$	$\frac{N^2}{4}(1 - \frac{37}{18N} + \frac{49}{18N^2})$
3	$\frac{577}{540}$	$\frac{N^3}{8}(1 - \frac{287}{90N} + \frac{407}{90N^2} - \frac{448}{135N^3})$
4	$\frac{1481}{864}$	$\frac{N^4}{16}(1 - \frac{20\,753}{5400N} + \frac{363\,091}{48\,600N^2} - \frac{17\,849}{2430N^3} + \frac{404}{81N^4})$
5	$\frac{28\,943}{19\,440}$	$\frac{N^5}{32}(1 - \frac{12\,533}{2625N} + \frac{765\,584\,3}{708\,750N^2} - \frac{131\,966\,6}{91\,125N^3} + \frac{380\,820\,31}{318\,937\,5N^4} - \frac{244\,912\,1}{425\,250N^5})$
6	$\frac{337\,444\,93}{158\,760\,00}$	$\frac{N^6}{64}(1 - \frac{363\,217\,1}{661\,500N} + \frac{351\,154\,798\,9}{238\,140\,000N^2} - \frac{211\,927\,785\,1}{893\,025\,00N^3} + \frac{614\,187\,878\,3}{238\,140\,000N^4} - \frac{624\,998\,186\,3}{357\,210\,000N^5} + \frac{954\,322\,601}{119\,070\,000N^6})$

TABLE II. LL coefficients for the amplitude $B(s)$.

# loops	$N = 3$	Arbitrary N
1	$\frac{2}{3}$	$\frac{2}{3}$
3	$\frac{181}{270}$	$\frac{N^2}{10} - \frac{203N}{1080} + \frac{361}{1080}$
5	$\frac{332\,074\,7}{510\,300\,0}$	$\frac{N^4}{56} - \frac{8609N^3}{126\,000} + \frac{892\,579N^2}{567\,000\,0} - \frac{173\,670\,61N}{102\,060\,000} + \frac{291\,493\,1}{204\,120\,00}$
7	$\frac{243\,374\,755\,934\,9}{378\,071\,064\,000\,0}$	$\frac{N^6}{288} - \frac{209\,635N^5}{106\,686\,72} + \frac{266\,068\,577\,671N^4}{448\,084\,224\,000\,0} - \frac{161\,665\,040\,906\,3N^3}{151\,228\,425\,600\,00} + \frac{163\,934\,003\,072\,87N^2}{120\,982\,740\,480\,000} - \frac{179\,751\,202\,247N}{172\,832\,486\,400\,0} + \frac{341\,420\,872\,829\,3}{604\,913\,702\,400\,00}$

lation gives $\mathcal{B}_0^{(0,0)(0,0)} = (N+8)/2$, which is nothing else as the coefficient of the 1-loop β-function in the ϕ^4 theory. For that case the solution of Eq. (16) is $\omega_{nj} = ((N+8)/2)^n \delta_{j0}$, that coincides with the solution for LLs [13] in the ϕ^4-theory which alternatively can be obtained by solving the 1-loop RG equations.

The nonlinear recursion relation (16) for LLs is valuable not only because it renders a breakthrough in the calculations of LLs, it also allows general theoretical studies of LLs anatomy in wide class of effective theories. As an example, let us consider the behavior of LLs in the large-N limit of $O(N+1)$ σ-model. In this limit we can neglect all terms in the expression (14) but the first one. Then, the recursion relation (16) for large N LLs is simplified considerably:

$$\omega_{nj}^{\mathrm{LN}} = \frac{1}{2n} \sum_{m=1}^{n-1} \frac{N}{2j+1} \omega_{mj}^{\mathrm{LN}} \omega_{(n-m-1)j}^{\mathrm{LN}}. \qquad (18)$$

The solution of this recursion is obvious:

$$\omega_{nj}^{\mathrm{LN}} = \left(\frac{N}{2}\right)^n \delta_{j0}. \qquad (19)$$

This solution is in agreement with the direct large-N calculations in the $O(N+1)$ σ-model, see, e.g., [14]. In order to compute the $1/N$ corrections to the leading result (19) we substitute the LLs in the form

$$\omega_{nj} = \left(\frac{N}{2}\right)^n \left[\delta_{j0} + \frac{c_{nj}^{(1)}}{N} + \ldots\right] \qquad (20)$$

into the recursion relation (16) and obtain the linear equation for the coefficients $c_{nk}^{(1)}$. This equation has a solution in terms of the Lerch function. The corresponding expression is rather long, instead we give the result for the LL coefficients A_n for the leading and subleading $1/N$ orders in the case of large number of loops $n \gg 1$. The leading $1/N$ asymptotic of the amplitude B we compute without any assumptions about n. The result is

$$A_n = \left(\frac{N}{2}\right)^n \left[1 - \left(\frac{\pi^2}{3} - 8(1 - \ln 2)\right)\frac{n}{N} + \ldots\right],$$
$$B_n = \left(\frac{N}{2}\right)^{n-1} \frac{2}{2+n} \left[1 + O\left(\frac{1}{N}\right)\right].$$

It is a remarkable result. It shows that the $1/N$ expansion for $O(N+1)$ σ-model fails in the chiral order $n \sim N$ and the expansion requires reordering.

In summary, we have developed the method of nonlinear recursion relations (16) which allows a calculation the leading chiral logarithms to essentially unlimited order. Furthermore, this method presents a puissant tool for study of general structure of infrared logarithms. It can be applied to any physics problem described by a nonrenormalizable effective low-energy Lagrangian, e.g., theory of critical phenomena, low-energy quantum gravity, theory of magnetics, etc.

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