Realization of Cohen-Glashow Very Special Relativity on Noncommutative Space-Time

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We show that the Cohen-Glashow very special relativity (VSR) theory [A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. **97**, 021601 (2006)] can be realized as the part of the Poincaré symmetry preserved on a noncommutative Moyal plane with lightlike noncommutativity. Moreover, we show that the three subgroups relevant to VSR can also be realized in the noncommutative space-time setting. For all of these three cases, the noncommutativity parameter $\theta^{\mu\nu}$ should be lightlike ($\theta^{\mu\nu}\theta_{\mu\nu} = 0$). We discuss some physical implications of this realization of the Cohen-Glashow VSR.

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The special theory of relativity postulates the Poincaré group as the symmetry of Nature. It is believed, however, that at very high energies the usual description of space-time in terms of a smooth manifold would break down and, together with it, the Lorentz invariance of physical theories. Various possible departures from Lorentz invariance at high energies have been studied, both theoretically and experimentally (see [1–3], and references therein). The problem addressed in Ref. [1] is whether Lorentz-invariant theories, like the standard model, could emerge as effective theories from a more fundamental scheme, perhaps operative at the Planck scale, which is invariant under very special relativity groups but not invariant under the full Poincaré group.

Very special relativity (VSR) has been introduced in Ref. [1] as symmetry under certain subgroups of the Poincaré group, which contain space-time translations and at least a 2-parametric proper subgroup of the Lorentz transformations, isomorphic to that generated by $K_x + J_y$ and $K_y - J_x$, where **J** and **K** are the generators of rotations and boosts, respectively. These subgroups of the Lorentz group share the remarkable property that, when supplemented with T, P, or CP, they will be enlarged to the full Lorentz group. This can be taken as the definition of VSR.

The requirement of energy-momentum conservation should be preserved in VSR; consequently, in all of its realizations the translational symmetry should be contained. Besides generators of translations P_μ, the minimal version of VSR includes the Abelian subgroup of the Lorentz group $T(2)$, generated by $T_1 = K_x + J_y$ and $T_2 = K_y - J_x$. The group $T(2)$ can be identified with the translation group on a two-dimensional plane. The other larger versions of VSR are obtained by adding one or two Lorentz generators to $T(2)$, which have a geometric realization on the two-dimensional plane: (i) $E(2)$, the group of two-dimensional Euclidean motion, generated by T_1, T_2, and J_z, with the structure:

$$[T_1, T_2] = 0, \qquad [J_z, T_1] = -iT_2, \qquad [J_z, T_2] = iT_1; \quad (1)$$

(ii) $HOM(2)$, the group of orientation-preserving similarity transformations, or homotheties, generated by T_1, T_2, and K_z, with the structure

$$[T_1, T_2] = 0, \qquad [T_1, K_z] = iT_1, \qquad [T_2, K_z] = iT_2; \quad (2)$$

(iii) $SIM(2)$, the group isomorphic to the four-parametric similitude group, generated by T_1, T_2, J_z, and K_z.

When attempting to construct a concrete realization of the VSR symmetry as a fundamental scheme within a "master theory," most certainly nonlocal [1] and which would lead in the low-energy limit to an effective Poincaré-invariant theory, one runs into the problem of the representation content of the master theory. The Lorentz subgroups involved in VSR have only one-dimensional representations, unlike the Lorentz group. The representations of $T(2)$, $E(2)$, $HOM(2)$, and $SIM(2)$ are automatically representations of the Lorentz group, but the reciprocal is not valid. As a result, if we construct the master theory based on the one-dimensional representations of the VSR subgroups, when requiring also P, T, or CP invariance, although the theory becomes invariant under the whole Lorentz group, the one-dimensional representations will not change. As a result, the effective theory would be doomed by its very poor representation content. Another possibility is to use in the realization of VSR the representations of the full Lorentz group but add a Lorentz-violating factor, such that the symmetry of the Lagrangian is reduced to one of the VSR subgroups of the Lorentz group. However, such an approach can hardly provide a *fundamental theory*, given that its symmetry does not match its representation content.

This contradiction can be resolved if we abandon the reasoning in terms of Lie groups or algebras and extend the discussion to (deformed) Hopf algebras. In the framework of Hopf algebras, there exist deformations which leave the structure of the algebra (commutation relation of the generators) untouched but affect other properties of the Hopf algebra, i.e., the coalgebra structure [4]. Since the commutation relations of generators are not deformed, it follows automatically that the Casimir operators are the same and the representation content of the deformed Hopf algebra is identical to the one of the undeformed algebra. On the other hand, the deformation of the coalgebra structure reduces the symmetry of the scheme. Such deformations are the twists introduced in Ref. [5], which turned out to provide a powerful concept, facilitating the systematic approach to deformation quantization [4].

Noncommutative quantum field theories (NC QFTs) are field theories constructed on space-times whose coordinates satisfy the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{3}$$

where θ can be a function of coordinates (with the condition that it satisfies the Jacobi identity). The commutation relations (3) usually spoil the Lorentz invariance (and sometimes also the translational invariance) of the NC QFTs; however, these theories remain invariant under the subgroup of the Poincaré group which preserves the covariance of (3) (see [6] for the case of constant θ).

The essential element for our discussion is that NC QFTs possess symmetry under various twisted Poincaré algebras, depending on the structure of θ [7,8] (see also [9]). The advantage of using the twisted Poincaré language for constructing physical theories is that, in spite of the lack of full Lorentz symmetry, the fields carry representations of the full Lorentz group [10,11] and the spin-statistics relation is still valid; the deformation then appears in the product of the fields (interaction terms).

Although we reviewed mainly the technical merits of the NC QFT as a candidate for realizing VSR, we should recall the main motivation for introducing the NC space-time, based on the interplay of quantum theory and classical gravity [12], as well as the emergence of NC QFTs as low-energy effective theories from string theory in Kalb-Ramond background field [13,14].

In this Letter, we show that the NC spaces with lightlike noncommutativity [15] offer a natural setting which realizes VSR, hence providing us with the well-studied theoretical setting of NC QFT, with twisted Poincaré symmetry, as a physical framework for VSR.

Space-time noncommutativity and VSR.—To start with, we focus on the NC spaces defined through (3). Since $\theta^{\mu\nu}$ is an antisymmetric two tensor, NC spaces can be classified according to the two Lorentz invariants

$$\Lambda^4 \equiv \theta_{\mu\nu}\theta^{\mu\nu}, \qquad L^4 \equiv \epsilon^{\alpha\beta\mu\nu}\theta_{\mu\nu}\theta_{\alpha\beta}. \tag{4}$$

Λ^4 is related to the noncommutativity scale, the scale where noncommutativity effects will become important, while L^4 is related to the smallest (space-time) volume that we can measure in a noncommutative theory.

Depending on whether L^4 and Λ^4 are positive, zero, or negative, one can recognize nine cases. The $L^4 \neq 0$ cases cannot be obtained as a decoupling (low-energy) limit of open string theory, do not lead to a unitary NC QFT theory [15], and hence are not usually considered. (However, the $\Lambda^4 = 0$, $L^4 \neq 0$ case is the famous Doplicher-Fredenhagen-Roberts [12] noncommutative space.)

For $L^4 = 0$, depending on the value of Λ^4, there are three types of noncommutative spaces: (i) $\Lambda^4 > 0$—spacelike (space-space) noncommutativity; (ii) $\Lambda^4 < 0$—timelike (time-space) noncommutativity; (iii) $\Lambda^4 = 0$—lightlike noncommutativity. When Λ is constant, for case (ii), it has been shown that there is no well-defined decoupled field theory limit for the corresponding open string theory [15]. In the field theory language, this shows itself as instability of the vacuum state and nonunitarity of the field theory on timelike NC space [16]. For the spacelike case (i) and lightlike case (iii), noncommutative field theory limits are well-defined and the corresponding field theories are perturbatively unitary.

Depending on the structure of the right-hand side of (3), there exist three types of NC deformations of the space-time which can be realized through twists of the Poincaré algebra [7–9]: (i) Constant $\theta^{\mu\nu}$—the Heisenberg-type commutation relations, defining the *Moyal space*:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{5}$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix. (ii) Linear $\theta^{\mu\nu}$, with the Lie-algebra type commutators:

$$[x^\mu, x^\nu] = iC^{\mu\nu}_\rho x^\rho, \tag{6}$$

describing an (associative but) noncommutative space if $C^{\mu\nu}_\rho$ are structure constants of an associative Lie algebra. (iii) *Quadratic noncommutativity*, the quantum group type of commutation relations:

$$[x^\mu, x^\nu] = \frac{1}{q} R^{\mu\nu}_{\rho\sigma} x^\rho x^\sigma. \tag{7}$$

All of the above-mentioned cases of noncommutative space-time have originally been studied in Ref. [17] with respect to the formulation of NC QFTs on those spaces. Only in case (i) is the translational invariance preserved in all of the directions of space-time. Since the translation symmetry is one of the requirements of the Cohen-Glashow VSR theories [1], only the Moyal NC space-time is relevant to VSR, and therefore here we mainly focus on the Moyal case. We shall briefly discuss the linear and quadratic $\theta^{\mu\nu}$ cases, since in special conditions, all of these types of noncommutativity can be put in a relation to certain Lorentz subgroups relevant to VSR.

$T(2)$ symmetry as lightlike noncommutativity.—Motivated by the above arguments, we set about finding a configuration of the antisymmetric matrix $\theta^{\mu\nu}$ which would be invariant under the $T(2)$ subgroup of the Lorentz group—the only of the VSR subgroups which admits invariant tensors, as also noted in [1]. If we denote the elements of the $T(2)$ subgroup by

$$\Lambda_1 = e^{i\alpha T_1} \quad \text{and} \quad \Lambda_2 = e^{i\beta T_2}, \tag{8}$$

the invariance condition for the tensor $\theta^{\mu\nu}$ is written as

$$\Lambda^\mu_{i\alpha}\Lambda^\nu_{i\beta}\theta^{\alpha\beta} = \theta^{\mu\nu}, \qquad i = 1, 2, \tag{9}$$

and infinitesimally

$$T^{\mu}_{i\alpha}\theta^{\alpha\nu} + T^{\nu}_{i\beta}\theta^{\mu\beta} = 0, \qquad i = 1, 2. \tag{10}$$

The nonvanishing elements of the matrix realizations of the generators T_1 and T_2 are (see, e.g., the monograph [18]): $(T_1)^0_1 = (T_1)^1_0 = (T_1)^1_3 = -(T_1)^3_1 = i$ and $(T_2)^0_2 = (T_2)^2_0 = (T_2)^2_3 = -(T_2)^3_2 = i$.

Plugging these values into (10), we find the solution

$$\theta^{0i} = -\theta^{3i}, \qquad i = 1, 2, \tag{11}$$

all of the other components of the antisymmetric matrix $\theta^{\mu\nu}$ being zero. Note that to obtain the above result we did not assume any special form for the x dependence of $\theta^{\mu\nu}$, and hence this holds for either of the three constant, linear, and quadratic cases.

With the above $\theta^{\mu\nu}$ we see that $\Lambda^4 = L^4 = 0$; that is, *a lightlike $\theta^{\mu\nu}$ is invariant under $T(2)$.*

One may use the light-cone frame coordinates

$$x^{\pm} = (t \pm x^3)/2, \qquad x^i, i = 1, 2. \tag{12}$$

In the above coordinate system, the only nonzero components of the lightlike noncommutativity (11) are $\theta^{-i} = \theta^{0i} = -\theta^{3i}$ (and $\theta^{+-} = \theta^{+i} = \theta^{ij} = 0$). In the light-cone coordinates (or light-cone gauge), one can take x^+ to be the light-cone time and x^- the light-cone space direction. In this frame, (light-cone) time commutes with the space coordinates. In the light-cone $(+, -, 1, 2)$ basis,

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & \theta' \\ 0 & -\theta & 0 & 0 \\ 0 & -\theta' & 0 & 0 \end{pmatrix}. \tag{13}$$

$E(2)$ and $SIM(2)$ invariant NC spaces.—A constant θ^{-i} breaks rotational invariance in the (x^1, x^2) plane, and hence larger VSR subgroups are not possible in the Moyal NC space case. The $E(2)$ invariant case can be realized in the linear, Lie-algebra-type noncommutative spaces, and $SIM(2)$ can be realized by quadratic noncommutativity.

The $E(2)$ case.—Recalling that J_z is the generator of rotations in the (x^1, x^2) plane while keeping $x^{\pm}$ invariant, and that x^i are invariant under K_z, which acts as scaling on x^- and x^+, the $E(2)$ case is realized when θ^{-i} is proportional to x^i and is independent of x^-. Noting that the only invariant tensors under J_z are δ_{ij} and ϵ_{ij}, then θ^{-i} has to be proportional to the product of either of these invariant tensors with the only available vector on the (x^1, x^2) plane:

$$[x^-, x^i] = i\ell\epsilon^{ij}x^j \quad \text{or} \tag{14a}$$

$$[x^-, x^i] = i\ell x^i. \tag{14b}$$

With the above choices, the translational symmetry along $x^{\pm}$ is preserved, while along x^i it is clearly lost.

Instead of x^i coordinates, we may work with the cylindrical coordinates on x^-, x^1, x^2 space, with the axis of the cylinder along x^-. If we denote the radial and angular coordinate on the (x^1, x^2) plane by ρ and ϕ, respectively, the case (14a) is then described by

$$[x^-, \rho] = 0, \qquad [\rho, e^{\pm i\phi}] = 0, \qquad [x^-, e^{\pm i\phi}] = \pm\lambda e^{\pm i\phi}. \tag{15}$$

This space is a collection of NC cylinders of various radii. There is a twisted Poincaré algebra which provides the symmetry for this case, while the other case cannot be generated by a twist [19]. In the above, ℓ and λ are deformation parameters of dimension length.

The $SIM(2)$ case.—From the above discussions it becomes clear that, to have both the K_z and J_z invariant noncommutative structures, we should take θ^{-i} linear in both x^- and x^i; therefore, the two possibilities are

$$[x^-, x^i] = i\frac{\xi - 1}{\xi + 1}\epsilon^{ij}\{x^-, x^j\} \quad \text{or} \tag{16a}$$

$$[x^-, x^i] = i\tan\xi\{x^-, x^i\}, \tag{16b}$$

preserving translational symmetry only along x^+ (where ξ is the dimensionless deformation parameter).

For neither of the above cases is there any twisted Poincaré algebra of the form discussed in Ref. [9] which provides these commutators [19]. The case (16b) in the above-mentioned cylindrical coordinates x^-, ρ, and ϕ takes the familiar form of a quantum (Manin) plane.

As mentioned above, the Cohen-Glashow VSR requires translation invariance, which is realized only in the constant $\theta^{\mu\nu}$ case; therefore, we continue with the discussion of QFTs on the lightlike Moyal plane, as the VSR-invariant theories. Further analysis of the linear and quadratic cases will be postponed to future works [19].

NC QFT on lightlike Moyal plane as VSR-invariant theory.—So far, we have shown that a Moyal plane with lightlike noncommutativity is invariant under the $T(2)$ VSR. Consequently, the NC QFTs constructed on this space possess also the same symmetry [6], as well as twisted Poincaré symmetry [7,8]. For any given QFT on commutative Minkowski space, its noncommutative counterpart, NC QFT, is obtained by replacing the usual product of functions (fields) with the nonlocal Moyal $*$ product (for a review on NC QFTs, see [20]):

$$(\phi * \psi)(x) = \phi(x)e^{i/2\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu}\psi(x). \tag{17}$$

Because of the twisted Poincaré symmetry, the fields carry representations of the full Lorentz group [10,11], but they admit transformations only under the stability group of lightlike $\theta^{\mu\nu}$, $T(2)$.

NC QFTs are CPT-invariant and satisfy the spin-statistics relation [21–23]. However, as shown in Ref. [21] for NC QED, C, P, and T symmetries are not individually preserved: For the time-space noncommutativity, which comprises also the lightlike case, P invariance requires also the transformation $\theta^{0i} \to -\theta^{0i}$, C invariance requires $\theta^{\mu\nu} \to -\theta^{\mu\nu}$, and T invariance requires $\theta^{ij} \to -\theta^{ij}$. Since $\theta^{\mu\nu}$ is invariant on the Moyal space, these transformations cannot occur. Consequently, requiring P, T, or CP invariance from NC QFT with $T(2)$ VSR symmetry is equivalent to taking $\theta \to 0$, in which case the emerging theory has full Poincaré symmetry, as predicted in [1].

Discussion and outlook.—We have shown that lightlike Moyal NC space provides a consistent framework for $T(2)$ Cohen-Glashow VSR-invariant theories. The other VSR groups $E(2)$, $SIM(2)$, and $HOM(2)$ are ruled out if the origin of Lorentz violation is in the NC structure of space-time, since the corresponding NC spaces are not translationally invariant. The realization of VSR as NC theories has several advantages. (i) Despite the lack of full Lorentz symmetry, one can still label fields by the Lorentz representations. For the NC QFTs we can rely on the basic notions of fermions and bosons, spin-statistics relation, and *CPT* theorem [21–23]. (ii) There is a simple recipe for constructing the NC version of any given QFT. Noncommutativity introduces a structure, fixing the form of the VSR-invariant action. (iii) In the NC setting we deal only with a single deformation parameter [the coordinates on the (x^1, x^2) plane can be chosen such that θ' in (13) is zero].

The parameter θ of the NC QFT realization of $T(2)$ VSR is of the dimension length-squared, and it defines the non-commutativity scale $\Lambda_{\rm NC} = 1/\sqrt{\theta}$. To find bounds on $\Lambda_{\rm NC}$ we need to compare results based on the NC models to the existing observations and data. These data can range from atomic spectroscopy and Lamb shift (see, e.g., [24]) to particle physics bounds on the electric-dipole moments of elementary particles. Since the structure of the terms involving the lightlike NC parameter is essentially the same as in the more-studied case with space-space non-commutativity [as it stems from the same $*$ product (17)], and based on explicit calculations [19], we can infer that the bounds on the NC parameter will be of the same order of magnitude as the previously calculated ones, i.e., $\Lambda_{\rm NC} > 10$ TeV [24] (see also [25] for a similar bound coming from clock-comparison experiments).

To construct a particle physics model based on the NC realization for VSR, we need to fix a lightlike $\theta^{\mu\nu}$ in any of the noncommutative models constructed so far, with generic noncommutativity. Although various basic features are common to all NC QFTs constructed with the $*$ product (17), the lightlike case has some specific features which are not shared by other NC QFTs on Moyal space-time. Moreover, the lightlike NC QFTs are also unitary field theories [15]. The construction of consistent lightlike NC models may lead to the derivation of more stringent bounds on $\Lambda_{\rm NC}$. The lightlike NC QFTs are expected to have many features in common with the space-space NC case, which has been thoroughly studied in the literature, and the main results of that case should also hold for the lightlike case. Clarifying which of the results carry over to the lightlike NC QFT will be discussed in a forthcoming communication [19].

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