Weakly Turbulent Magnetohydrodynamic Waves in Compressible Low-β Plasmas

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In this Letter, weak-turbulence theory is used to investigate interactions among Alfvén waves and fast and slow magnetosonic waves in collisionless low-β plasmas. The wave kinetic equations are derived from the equations of magnetohydrodynamics, and extra terms are then added to model collisionless damping. These equations are used to provide a quantitative description of a variety of nonlinear processes, including parallel and perpendicular energy cascade, energy transfer between wave types, "phase mixing," and the generation of backscattered Alfvén waves.

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Turbulence at length scales smaller than the collisional mean free path $\lambda_{\rm mfp}$ plays a central role in a wide range of astrophysical and laboratory plasmas. In general, the analysis of waves and turbulence at scales $<\lambda_{\rm mfp}$ requires the use of kinetic theory. However, in some cases fluid models are approximately valid even at such collisionless scales. For example, if $\beta = 8\pi p/B^2 \ll 1$, where p is the pressure and $\boldsymbol{B}$ is the magnetic field, then magnetohydrodynamics (MHD) provides an approximately correct description of the fast magnetosonic wave ("fast wave") when $\lambda < \lambda_{\rm mfp}$ and $\omega \ll \Omega_i$, where λ is the wavelength and Ω_i is the proton cyclotron frequency [1]. Similarly, MHD accurately describes both Alfvén waves and anisotropic Alfvén-wave turbulence when $r_i \ll \lambda < \lambda_{\rm mfp}$ and $\omega \ll \Omega_i$, where r_i is the proton gyroradius [1,2]. MHD is approximately accurate in these cases because the dynamics are governed primarily by magnetic forces and inertia, while the pressure tensor and collisionless damping play only a minor role. In this Letter, MHD is used to model turbulence at length scales $\gg r_i$ and $<\lambda_{\rm mfp}$ and frequencies $\ll \Omega_i$ in low-β plasmas. To account for the strong collisionless damping of slow magnetosonic waves and the weak collisionless damping of fast waves [1], extra damping terms are added to the equations for the wave power spectra. Although this approach is only an approximation to the full kinetic behavior of the plasma, the comparative simplicity of MHD makes it possible to describe the physics within the MHD model in great detail and thereby gain useful insight into the full problem.

The basic phenomenology of MHD turbulence depends on whether the turbulence is weak or strong, which in turn depends on the value of $\omega_k \tau_k$, where ω_k is the linear wave frequency at wave vector $\boldsymbol{k}$ and τ_k is the time scale on which the fluctuations at wave vector $\boldsymbol{k}$ evolve due to nonlinearities. If $|\omega_k|\tau_k \gg 1$, then the turbulence is weak, the fluctuations can be approximated as a collection of small-amplitude waves, and the interactions between waves can be analyzed using perturbation theory [3,4]. On the other hand, if $|\omega_k|\tau_k \lesssim 1$, then the fluctuations are not wavelike and the turbulence is strong. In MHD, τ_k is at least as large as $\sim(k\delta v_k)^{-1}$, where δv_k is the rms amplitude of the velocity fluctuation at scale k^{-1}. Thus, the condition $|\omega_k|\tau_k \gg 1$ is satisfied provided $|\omega_k| \gg k\delta v_k$.

An important point is that the weak and strong turbulence limits can apply to different components of the turbulence within a single plasma [5–7]. For Alfvén waves, $\omega_k = \pm k_z v_A$, where $v_A = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén speed, ρ_0 is the background density, and $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$ is the background magnetic field. As a result, Alfvén-wave turbulence is strong for sufficiently small $|k_z|/k_\perp$, where $\boldsymbol{k}_\perp = \boldsymbol{k} - k_z\hat{\boldsymbol{z}}$. On the other hand, Alfvén waves with $|k_z| \gtrsim k_\perp$ and $\delta v_k \ll v_A$ are weakly turbulent. Similarly, fast waves satisfy $\omega_k \simeq \pm k v_A$ in low-β plasmas, and are thus weakly turbulent provided $\delta v_k \ll v_A$. This Letter focuses on weak turbulence, but a method to account for strong-Alfvén-wave turbulence is also described.

The equations of ideal MHD are

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0, \tag{1}$$

$$\rho\left(\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right) = -\boldsymbol{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}, \tag{2}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}), \tag{3}$$

where ρ is the density and $\boldsymbol{v}$ is the velocity. The specific entropy $[\propto \ln(p\rho^{-\gamma})]$ is taken to be a constant (where γ is the ratio of specific heats). Each fluid quantity is taken to be the sum of a uniform background value plus a small-amplitude fluctuation: $\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{B}$, $p = p_0 + \delta p$, $\rho = \rho_0 + \delta\rho$, and $\boldsymbol{v} = \boldsymbol{v}_0 + \delta\boldsymbol{v}$, with $v_0 \ll \delta v \ll v_A$. The (spatial) Fourier transforms of $\delta\boldsymbol{v}$ and $\boldsymbol{b} \equiv \delta\boldsymbol{B}/\sqrt{4\pi\rho_0}$ can be written as

$$\boldsymbol{v}_k = v_{a,k}\hat{\boldsymbol{e}}_{a,k} + v_{f,k}\hat{\boldsymbol{k}}_\perp + v_{z,k}\hat{\boldsymbol{z}},$$

$$\boldsymbol{b}_k = b_{a,k}\hat{\boldsymbol{e}}_{a,k} + b_{f,k}\hat{\boldsymbol{e}}_{f,k},$$

where $\hat{\boldsymbol{e}}_{a,k} = \hat{\boldsymbol{z}}\times\hat{\boldsymbol{k}}_\perp$, $\hat{\boldsymbol{k}}_\perp = \boldsymbol{k}_\perp/k_\perp$, and $\hat{\boldsymbol{e}}_{f,k} = \hat{\boldsymbol{e}}_{a,k}\times\boldsymbol{k}/k$. The Alfvén-wave amplitudes at wave vector $\boldsymbol{k}$ are

$$a_k^\pm = \frac{1}{\sqrt{2}}(v_{a,k} \mp b_{a,k}).$$

The fast and slow-wave amplitudes, $f_k^\pm$ and $s_k^\pm$, are given by

$$\boldsymbol{w} = 2^{-1/2}\boldsymbol{M}\cdot\boldsymbol{u},$$

where $\boldsymbol{w} = (f_k^+, f_k^-, s_k^+, s_k^-)$, $\boldsymbol{u} = (h_k, v_{f,k}, v_{z,k}, b_{f,k})$, $h_k = c_s\rho_k/\rho_0$, $c_s = \sqrt{\gamma p_0/\rho_0}$ is the sound speed, and ρ_k is the Fourier transform of $\delta\rho$. The matrix $\boldsymbol{M}$ is an infinite series in powers of $\varepsilon = c_s/v_A = \sqrt{\gamma\beta/2}$. To order ε^2,

$$\boldsymbol{M} = \begin{pmatrix} \varepsilon\eta & 1 & \varepsilon^2\eta\mu & -1+\varepsilon^2\eta^2/2 \\ -\varepsilon\eta & 1 & \varepsilon^2\eta\mu & 1-\varepsilon^2\eta^2/2 \\ 1-\varepsilon^2\eta^2/2 & -\varepsilon^2\eta\mu & 1 & \varepsilon\eta \\ -1+\varepsilon^2\eta^2/2 & -\varepsilon^2\eta\mu & 1 & -\varepsilon\eta \end{pmatrix},$$

where $\mu = \cos\theta$, $\eta = \sin\theta$, and θ is the angle between $\boldsymbol{k}$ and $\hat{\boldsymbol{z}}$. The Fourier transforms of Eqs. (1) through (3), expressed in terms of $s_k^\pm$, $a_k^\pm$, and $f_k^\pm$, become

$$\frac{\partial s_k^\pm}{\partial t} + i\omega_{s,k}^\pm s_k^\pm = N_{s,k}^\pm, \tag{4}$$

$$\frac{\partial a_k^\pm}{\partial t} + i\omega_{a,k}^\pm a_k^\pm = N_{a,k}^\pm, \tag{5}$$

$$\frac{\partial f_k^\pm}{\partial t} + i\omega_{f,k}^\pm f_k^\pm = N_{f,k}^\pm, \tag{6}$$

where the right-hand sides are the nonlinear terms, $\omega_{a,k}^\pm = \pm k_z v_A$, and, to lowest order in ε, $\omega_{s,k}^\pm = \pm k_z c_s$ and $\omega_{f,k}^\pm = \pm k v_A$.

The power spectra are defined by the equations $\langle s_k^\pm (s_{k1}^\pm)^\star\rangle = S_k^\pm \delta(\boldsymbol{k}-\boldsymbol{k}_1)$, $\langle a_k^\pm (a_{k1}^\pm)^\star\rangle = A_k^\pm \delta(\boldsymbol{k}-\boldsymbol{k}_1)$, and $\langle f_k^+ (f_{k1}^+)^\star\rangle = F_k \delta(\boldsymbol{k}-\boldsymbol{k}_1)$, where $\langle\ldots\rangle$ denotes an ensemble average. The quantity $A_k^\pm$ ($S_k^\pm$) is proportional to the energy per unit volume in k space of Alfvén waves (slow waves) propagating in the $\pm z$ direction. The quantity F_k is proportional to the energy per unit volume in k space of fast waves propagating in the $\boldsymbol{k}$ direction. Cylindrical symmetry about the z axis is assumed, so that $S_k^\pm = S^\pm(k_\perp, k_z, t)$, $A_k^\pm = A^\pm(k_\perp, k_z, t)$ and $F_k = F(k_\perp, k_z, t)$.

In the weak-turbulence limit, the wave kinetic equations can be obtained from Eqs. (4) through (6) using the standard techniques of [3,4]. These equations express $\partial S_k^\pm/\partial t$, $\partial A_k^\pm/\partial t$, and $\partial F_k/\partial t$ as series in powers of ε. As written below, the lowest-order terms in these series are $\propto \varepsilon^{-2}$, contain $S_k^\pm$, and are associated with the slow-wave density fluctuation, $\rho_k \simeq k_z v_{z,k}\rho_0/\omega_{s,k}^\pm$, which is a factor $\varepsilon^{-1}\csc\theta$ larger than the fast-wave density fluctuation, $\rho_k \simeq k_\perp v_{f,k}\rho_0/\omega_{f,k}^\pm$, when $v_{z,k} = v_{f,k}$ [8]. Although proportional to ε^{-2}, these terms may nevertheless be small, because strong collisionless damping [1] makes $S_k^\pm$ much smaller than $A_k^\pm$ and F_k. In this Letter, the $\varepsilon^{-2}S_k^\pm$ terms are retained, but the nonlinear terms containing $S_k^\pm$ at higher order in ε are dropped, with the exception of the $\delta(q_z)$ term in Eq. (7), which is retained for reasons discussed below. Of the terms that do not contain $S_k^\pm$, only the leading-order terms ($\propto\varepsilon^0$) are kept. The wave kinetic equations then become

$$\frac{\partial S_k^\pm}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})[\delta(q_z)4k_\perp^2\bar{m}^2(A_q^+ + A_q^-)(S_p^\pm - S_k^\pm) + \delta(p-q)k_z^2 l^2 F_p F_{-q} + \delta(p_z-q_z)k_z^2 l^2 A_p^+ A_q^-$$
$$+ \delta(p_z+q)k_z^2\bar{l}^2(A_p^+ F_q + A_p^- F_{-q}) + \delta(p_z-q)k_z^2\bar{l}^2(A_p^+ F_{-q} + A_p^- F_q)] - 2\gamma_{s,k}^\pm S_k^\pm, \tag{7}$$

$$\frac{\partial A_k^+}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{\delta(q_z)8(k_\perp n\bar{m})^2 A_q^-(A_p^+ - A_k^+) + \delta(k_z+p_z+q)k_z\Lambda_{q-pk}(k_z A_p^- F_{-q} + p_z F_{-q}A_k^+ + qA_p^- A_k^+)$$
$$+ \delta(k_z+p_z-q)k_z\Lambda_{q-pk}(k_z A_p^- F_q + p_z F_q A_k^+ - qA_p^- A_k^+) + \delta(k_z-p+q)k_z M_{pk-q}(k_z F_p F_{-q} - pF_{-q}A_k^+ + qF_p A_k^+)$$
$$+ \delta(q-k_z)p_z A_k^+\left[2(k_z+p_z)F_q + p_z q\frac{\partial F_q}{\partial q}\right] + \delta(q+k_z)p_z A_k^+\left[2(k_z+p_z)F_{-q} + p_z q\frac{\partial F_{-q}}{\partial q}\right]$$
$$+ \varepsilon^{-2}k_z^2(S_p^+ + S_p^-)[\delta(q-k_z)\bar{m}^2(F_q - A_k^+) + \delta(q+k_z)\bar{m}^2(F_{-q} - A_k^+) + \delta(p_z)m^2(A_q^+ - A_k^+)$$
$$+ \delta(k_z+q_z)m^2(A_q^- - A_k^+)] + \delta(q_z+k_z)4k_z^2 A_k^+\frac{\partial}{\partial q_z}(q_z A_q^-)\Big\} - 2\gamma_{a,k}^+ A_k^+, \tag{8}$$

$$\frac{\partial F_k}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{9\sin^2\theta[\delta(k-p-q)kqF_p(F_q - F_k) + \delta(k+p-q)k(kF_{-p}F_q + pF_q F_k - qF_{-p}F_k)]$$
$$+ \delta(k-p_z+q_z)k\Lambda_{kpq}(kA_p^+ A_q^- - p_z A_q^- F_k + q_z A_p^+ F_k) + \delta(k-p_z-q)kM_{kpq}(kA_p^+ F_q - p_z F_q F_k - qA_p^+ F_k)$$
$$+ \delta(k+p_z-q)kM_{-k-p-q}(kA_p^- F_q + p_z F_q F_k - qA_p^- F_k) + \delta(k-q)k^{-3}p_z F_k\left[k_z\frac{\partial}{\partial q}(q^4 F_q) - k^2 q_z\frac{\partial}{\partial q}(q^2 F_q)\right]$$
$$+ \varepsilon^{-2}k^2(S_p^+ + S_p^-)[\delta(k-q)m^2(F_q - F_k) + \delta(k-q_z)\bar{m}^2(A_q^+ - F_k) + \delta(k+q_z)\bar{m}^2(A_q^- - F_k)]$$
$$+ \delta(k-q_z)p_z F_k\left(2k_z A_q^+ + kp_z\frac{\partial A_q^+}{\partial q_z}\right) + \delta(k+q_z)p_z F_k\left(2k_z A_q^- - kp_z\frac{\partial A_q^-}{\partial q_z}\right)\Big\} - 2\gamma_{f,k}F_k, \tag{9}$$

where $\Lambda_{kpq} = \Lambda(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}) = k^{-2}(k_\perp l + 2p_\perp m + 2q_\perp n)^2$, $M_{kpq} = M(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}) = k^{-2}[k_\perp \bar{l} + p_\perp(\cos\alpha - 1)\bar{m} + k(\sin\alpha)\bar{n}]^2$, α is the angle between $\hat{z}$ and $\boldsymbol{q}$, $F_{-q} = F(q_\perp, -q_z, t)$, and $\gamma^\pm_{s,k}$, $\gamma^\pm_{a,k}$, and $\gamma_{f,k}$ are the linear damping rates. The partial derivative $\partial F_q/\partial q$ is taken at constant α, and the partial derivative $\partial A_q/\partial q_z$ is taken at constant $q_\perp$. In the triangle with sides of lengths $k_\perp$, $p_\perp$, and $q_\perp$, the interior angles opposite the sides of length $k_\perp$, $p_\perp$, and $q_\perp$ are denoted σ_k, σ_p, and σ_q, and $l = \cos\sigma_k$, $m = \cos\sigma_p$, $n = \cos\sigma_q$, $\bar{l} = \sin\sigma_k$, $\bar{m} = \sin\sigma_p$, and $\bar{n} = \sin\sigma_q$. The equation for $\partial A^-_k/\partial t$ is obtained by setting $A^\pm_k \to A^\mp_k$, $F_k \to F_{-k}$, $S^\pm_k \to S^\mp_k$, and $\gamma^+_{a,k} \to \gamma^-_{a,k}$ in Eq. (8).

The "collision integrals" on the right-hand sides of Eqs. (7) through (9) represent the effects of resonant three-wave interactions and sum over all wave number triads involving $\boldsymbol{k}$ that satisfy the resonance conditions $\boldsymbol{k} = \boldsymbol{p} + \boldsymbol{q}$ and $\omega_k = \omega_p + \omega_q$, where ω_k is the frequency at wave number $\boldsymbol{k}$. When $A^+_k = 0$ at some wave vector $\boldsymbol{k}_1$, the only nonvanishing terms in $\partial A^+_k/\partial t$ are non-negative at $\boldsymbol{k}_1$. Analogous statements hold for A^-_k, $S^\pm_k$ and F_k. Equations (7) through (9) thus ensure that the spectra remain non-negative. When the linear damping terms are dropped, Eqs. (7) through (9) conserve the energy per unit mass $\int d^3k(A^+_k + A^-_k + 2F_k + S^+_k + S^-_k)/2$ and the pseudo-momentum $\int d^3k[A^+_k - A^-_k + 2(\cos\theta)F_k + \varepsilon^{-1}(S^+_k - S^-_k)]/(2v_A)$. When the equation for $\partial\langle v_z\rangle/\partial t$ is taken into account, it can be shown that resonant three-wave interactions also conserve the cross helicity $\langle \boldsymbol{v}\cdot\boldsymbol{B}\rangle$ and momentum $\langle \rho v_z\rangle$.

At $k_z = 0$, the only nonzero term in the collision integral in Eq. (8) is the term proportional to $\delta(q_z)$. This term represents interactions between three Alfvén waves ("AAA interactions"), which transfer Alfvén-wave energy at all k_z to larger $k_\perp$ but not towards larger $|k_z|$ [9–14]. In AAA interactions, each Alfvén-wave type (a^+ or a^-) is cascaded by the other Alfvén-wave type. Thus, if $A^\mp(k_\perp, 0) = 0$ [where $A^\mp(k_\perp, 0)$ denotes $A^\mp_k$ evaluated at $k_z = 0$], then the AAA term in $\partial A^\pm_k/\partial t$ vanishes. A Zakharov transformation can be used to show that $A^\pm(k_\perp, 0) \propto k_\perp^{-n^\pm}$ is a steady-state solution to Eq. (8) for $k_z = 0$ in the absence of dissipation, provided $n^+ + n^- = 6$, as in the incompressible case [12]. When dissipation is included, these power laws become approximate solutions for $A^\pm(k_\perp, 0)$ within the inertial range. The Alfvén-wave spectra at $k_z = 0$ are not affected by the value of $A^\pm_k$ at nonzero k_z or by the slow-wave or fast-wave spectra.

At $k_z = 0$, the only nonzero term on the right-hand side of Eq. (7) is the term $\propto \delta(q_z)$, which represents the mixing of slow waves by Alfvén waves, which transfers slow-wave energy to larger $k_\perp$ but not to larger $|k_z|$. This term is identical to the expression describing the mixing of a passive scalar by weak Alfvén-wave turbulence, with $S^\pm_k$ replacing the passive-scalar spectrum. In the "imbalanced" case in which $A^+(k_\perp, 0) \gg A^-(k_\perp, 0)$ within the inertial range, the quantity $(A^+_q + A^-_q)$ in this "passive-scalar mixing term" can be approximated as simply A^+_q. If $A^+(k_\perp, 0) \propto k_\perp^{-n^+}$, a Zakharov transformation can then be used to show that $S^\pm(k_\perp, 0) \propto k_\perp^{-6+n^+}$ is a steady-state solution to Eq. (7) at $k_z = 0$ in the absence of dissipation. Thus, the slow-wave spectrum at $k_z = 0$ (and hence also the spectrum of a passive scalar) mimics the spectrum of the minority Alfvén-wave type, $A^-(k_\perp, 0)$. Although all other terms in the wave kinetic equations containing $S^\pm_k$ at orders higher than ε^{-2} have been discarded, the $\delta(q_z)S^\pm_k$ term in Eq. (7) has been retained because it can dominate as $k_z \to 0$, since the other nonlinear terms and the linear (Landau) damping term vanish in this limit. [Because collisionless damping keeps $S^\pm_k$ small at nonzero k_z, the cascade of slow-wave energy to larger $|k_z|$ arising from interactions among slow waves is neglected in Eq. (7).]

The term $\propto \varepsilon^{-2}\delta(p_z)$ in Eq. (8) represents "phase-mixing." Slow-wave density fluctuations at $k_z = 0$ cause the Alfvén speed to vary in the directions perpendicular to $\boldsymbol{B}_0$. As a result, Alfvén-wave phase fronts travel at different speeds on different field lines, transferring Alfvén-wave energy to larger $k_\perp$ [15]. (Density fluctuations at $k_z = 0$ associated with passive-scalar entropy waves would have the same effect.) Phase mixing and AAA interactions both cause a perpendicular cascade of Alfvén-wave energy. The relative strength of these two processes varies with θ, with the relative importance of phase mixing increasing as $|k_z|/k_\perp$ increases.

In Eq. (9), the terms proportional to $9\sin^2\theta$ represent interactions between three fast waves ("FFF interactions"). The FFF terms are the same as the collision integral for weak acoustic turbulence [4], up to an overall multiplicative factor proportional to $\sin^2\theta$. As $\sin\theta \to 0$, the acousticlike FFF interactions weaken because the fast waves become less compressive [16]. Energy is transferred from small k to large k by FFF interactions [6,16]. The resonance conditions for FFF interactions require that $\boldsymbol{p}$ and $\boldsymbol{q}$ be parallel or anti-parallel to $\boldsymbol{k}$, indicating that FFF interactions transfer energy along radial lines in k-space [6,16]. The terms containing M_{kpq} in Eqs. (8) and (9) represent interactions between one Alfvén wave and two fast waves ("AFF interactions"). When $k_\perp \ll |k_z|$, AFF interactions cause $A^\pm(k_\perp, k_z)$ to become approximately equal to $F(k_\perp, \pm|k_z|)$ [16]. The combination of FFF and AFF interactions results in a "parallel cascade", i.e., a transfer of Alfvén-wave and fast-wave energy to larger $|k_z|$ [16].

The $\varepsilon^{-2}S^\pm_p$ terms in Eq. (9) represent the "resonant scattering" of fast waves into either new fast waves or Alfvén waves of equal frequency but different wavenumber [8]. The $\varepsilon^{-2}\delta(k - q)$ term in Eq. (9) acts to isotropize F_k. The $\varepsilon^{-2}S^\pm_p$ terms in Eq. (8) other than the "phase-mixing" term are also denoted "resonant-scattering" terms, and represent the conversion of an Alfvén wave into a new Alfvén wave or fast wave of equal frequency. If $S^\pm_k \gtrsim A^\pm_k$ and $S^\pm_k \gtrsim F^\pm_k$, then resonant scattering and phase mixing are the most rapid nonlinear processes in the $\beta \to 0$ limit [8]. On the other hand, in collisionless systems, Landau damping can reduce $S_k^\pm$ sufficiently that resonant-scattering is weak. (Phase mixing involves $S_k^\pm$ at $k_z = 0$ where Landau damping vanishes and thus can be very efficient even in collisionless systems.)

The "resonant-scattering" term in Eq. (8) $\propto \varepsilon^{-2}\delta(k_z + q_z)(S_p^+ + S_p^-)$ represents the interaction of a slow wave with an Alfvén wave traveling in one direction along the magnetic field to produce an Alfvén wave traveling in the opposite direction. This generation of "back-scattered" Alfvén waves does not produce waves at $k_z = 0$ and thus does not contribute to AAA interactions or the associated perpendicular cascade of Alfvén-wave energy. Although the (hypothetical) conversion of A^- energy into A^+ energy would violate cross-helicity conservation in incompressible MHD, the generation of back-scattered Alfvén waves in compressible MHD does conserve cross helicity when one takes into account the associated change in the average flow velocity $\langle v_z \rangle$.

The $\delta(k - q)$ term in Eq. (9) that does not contain $S_p^\pm$ represents the generation of slow waves by fast waves. If the wave fields are viewed as the sum of wave quanta, each of energy $\hbar|\omega_k|$ and momentum $\hbar \boldsymbol{k}$, then this $\delta(k - q)$ term represents the process $f \to f + s$, i.e., a fast-wave decaying into a slow wave and a new fast wave. This $\delta(k - q)$ term conserves the total number of fast-wave quanta $N_f = \int d^3k[F_k/(\hbar|\omega_{f,k}^\mp|)]$, but decreases the fast-wave energy $E_f = \int d^3k F_k$, and thus causes an inverse cascade of fast-wave quanta to smaller frequency, i.e., a decrease in the average fast-wave frequency $\bar{\omega}_f \equiv E_f/\hbar N_f$. The energy drained from fast waves is transferred to slow waves [through the $\delta(p - q)$ term in Eq. (7)], which are then rapidly damped.

The term $(\pi/4v_A)\int d^3p\, d^3q\, \delta(\boldsymbol{k} - \boldsymbol{p} - \boldsymbol{q})\delta(q_z + k_z)4k_z^2 A_k^+(\partial/\partial q_z)(q_z A_q^-)$ in Eq. (8) is denoted I_k^+, and the corresponding term in the equation for $\partial A_k^-/\partial t$ is denoted I_k^-. These terms represent the generation of slow waves by the interaction of oppositely directed Alfvén waves, i.e. $a^\pm \to a^\mp + s$. Upon defining $E_k^\pm = \int dk_x dk_y A_k^\pm$ and $Q_k = \pi k_z^2 E_k^+ E_k^-/v_A$, one can show that $H_k \equiv \int dk_x dk_y (I_k^+ + I_k^-) = -Q_k + (\partial/\partial k_z)(k_z Q_k)$, where the $(\partial/\partial k_z)(k_z Q_k)$ term represents a flux of Alfvén-wave energy to smaller $|k_z|$ (inverse cascade). The energy drained from Alfvén waves via the $-Q_k$ term in H_k is transferred to slow waves through the $\delta(p_z - q_z)$ term in Eq. (7), which then undergo rapid ion Landau damping [17]. This mechanism for transferring Alfvén-wave energy to the ions is weak for "quasi-2D" fluctuations with $|k_z| \ll k_\perp$ because of the factor of k_z^2 in $I_k^\pm$.

Equation (8) can be modified to allow for the possibility of strong-Alfvén-wave turbulence at small $|k_z|$ by replacing the AAA term [$\propto \delta(q_z)$] in Eq. (8) with the advection and diffusion terms on the right-hand side of Eq. (15) of [18] multiplied by a factor of 2 to convert to the normalization of $A_k^\pm$ used in this Letter. Similar generalizations are possible for the "phase-mixing" and "passive-scalar mixing" terms.

The interplay between the various nonlinear processes described in this Letter depends upon the value of β as well as the amplitudes and anisotropies of the different wave types at the forcing scale or "outer scale." For example, greater excitation of Alfvén waves with $|k_z| \gtrsim k_\perp$ and fast-waves strengthens the parallel cascade. (Alfvén waves at $k_\perp \gg |k_z|$ cause only a weak secondary excitation of the fast waves and Alfvén waves with $|k_z| > k_\perp$ that participate in the parallel cascade.) On the other hand, the perpendicular cascade is strengthened by increasing the excitation at $k_z = 0$ of $S_k^\pm$, entropy waves, and both A_k^+ and A_k^-. A stronger perpendicular cascade then weakens the parallel cascade by draining energy out of the "quasi-parallel" region of k space in which $|k_z| > k_\perp$, reducing the amount of wave energy that reaches very large $|k_z|$. Numerical solutions to Eqs. (7) through (9) will be useful for describing turbulence in settings such as the solar corona, solar flares, and Earth's magnetosphere.

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