Magnetization and Spin Excitations of Non-Abelian Quantum Hall States

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Significant insights into non-Abelian quantum Hall states are obtained from studying special multi-particle interaction Hamiltonians, whose unique ground states are the Moore-Read and Read-Rezayi states for the case of spinless electrons. We generalize this approach to include the electronic spin-1/2 degree of freedom. We demonstrate that in the absence of Zeeman splitting, the ground states of such Hamiltonians have large degeneracies and very rich spin structures. The spin structure of the ground states and low-energy excitations can be understood based on an emergent SU(3) symmetry for the case corresponding to the Moore-Read state. These states with different spin quantum numbers represent non-Abelian quantum Hall states with different magnetizations, whose quasihole properties are likely to be similar to those of their spin-polarized counterparts.

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The possibility of quantum Hall states with fractionally charged quasiparticles that obey non-Abelian statistics has attracted tremendous interest recently [1–3], partly because of the potential of using these non-Abelian quasiparticles for quantum information storage and processing in an intrinsically fault-tolerant fashion [4–7]. Among such non-Abelian quantum Hall states, the most studied are the Moore-Read (MR) state [8], which may have been realized at Landau level (LL) filling factor $\nu = 5/2$ [9], and the Read-Rezayi (RR) states [10], which may have been realized at $\nu = 12/5$ [11] for the case of level $k = 3$ (see below for a definition). In these states the spins of the electrons occupying the valence Landau level (which in experimental systems is the first excited Landau level) are assumed to be fully polarized. However, this is an assumption which has not been fully tested numerically. The only exception is for the case of $\nu = 5/2$ where Morf [12] showed that the fully polarized state (which has a large overlap with the MR state) has lower energy than the spin singlet state, consistent with a more recent work using Monte Carlo simulation to evaluate the energies of the MR and spin-unpolarized 331 states [13]; all other numerical studies [14–16] *assume* full polarization. This is very unsatisfactory because, in typical systems, the Zeeman splitting due to electron spin is smaller than the Coulomb energy scale by about 2 orders of magnitude. The situation started to change only very recently since Feiguin *et al.* [17] carefully studied the magnetization of a half-filled first excited LL and found compelling evidence that suggests the electron spins are fully polarized for the case of Coulomb interaction, even in the *absence* of Zeeman splitting. Experimentally, attempts to detect spin polarization at $\nu = 5/2$ are on-going and remain inconclusive at this point [18].

In the present Letter we take an approach that is different but complementary to that of Ref. [17] and study the case of a special three-body interaction [19] that makes the MR state the unique ground state for spin-polarized electrons at half-filling. The special properties of this interaction allow us to establish a number of exact results. When applied to the case of spin-1/2 electrons (without Zeeman splitting), we find that a large ground state degeneracy appears with *different* total spin quantum numbers. These degenerate ground states are constructed explicitly and they form a single SU(3) multiplet. Such constructions can be generalized to the RR states when spin is included. This suggests that this family of non-Abelian quantum Hall states may have very rich spin structure. We further present numerical evidence suggesting that the low-energy spectrum of the system is consistent with an emergent SU(3) symmetry in the long-wavelength and low-energy limit for the MR case.

The three-body interaction that makes the MR state the exact ground state at half-filling takes the form:

$$H_{3B} = \sum_{i<j<k} S_{ijk}[\nabla_i^2 \nabla_j^4 \delta(\mathbf{r}_i - \mathbf{r}_j)\delta(\mathbf{r}_i - \mathbf{r}_k)], \qquad (1)$$

where S is a symmetrizer: $S_{123}[f_{123}] = f_{123} + f_{231} + f_{312}$, and f is symmetric in its first two indices. For spinless (or spin-polarized) electrons, the following MR state is the unique zero-energy ground state at half-filling:

$$\psi_{\rm MR} = \left[\prod_{i<j}(z_i - z_j)^2\right] A\left(\frac{1}{z_1 - z_2} \cdots \frac{1}{z_{2N-1} - z_{2N}}\right), \qquad (2)$$

where A is the antisymmetrizer, N is the number of pairs (so we have $N_e = 2N$ electrons), and we neglected the common exponential factor of LL wave functions. $\psi_{\rm MR}$ is annihilated by H_{3B} because it vanishes sufficiently fast as three particle coordinates approach each other. We now generalize $\psi_{\rm MR}$ to include spin degrees of freedom and construct the following zero-energy states in which we keep the Jastrow factor $[\prod_{i<j}(z_i - z_j)^2]$ of Eq. (2) while we modify the Pfaffian factor $A(\cdots)$:

$$\psi(z_1, \chi_1; \cdots; z_{2N}, \chi_{2N}) = \left[\prod_{i<j}(z_i - z_j)^2\right] A\left[\left(\frac{1}{z_1 - z_2}\cdots\frac{1}{z_{2N-1} - z_{2N}}\right)\left(\sum_{\{\chi\}} c_{\{\chi\}}\chi_{12}\cdots\chi_{2N-1,2N}\right)\right], \tag{3}$$

where χ_i is the spin-wave function of electron i and χ_{ij} is the spin-wave function of the pair made up of electrons i and j. Obviously Eq. (3) reduces to Eq. (2) when we take $\chi_{ij} = |\uparrow\rangle_i|\uparrow\rangle_j$, so that the electron spins are fully polarized. Also because the orbital part of (3) has the same asymptotic behavior as (2) when 3 electrons approach each other, (3) is also annihilated by H_{3B}.

We now consider the constraint on $c_{\{\chi\}}$ imposed by the antisymmetrizer A. Because of the fact that the orbital part is antisymmetric under the exchange between z_{2j-1} and z_{2j}, $\chi_{2j-1,2j}$ must be *symmetric* under such exchange; i.e., $\chi_{2j-1,2j}$ must represent a triplet state formed by electrons $2j-1$ and $2j$. Furthermore, $c_{\{\chi\}}$ must be symmetric under the exchange of different pairs $(2j-1, 2j)$ and $(2k-1, 2k)$; as a result $c_{\{\chi\}}$ represents a *totally symmetric* spin state formed by N spin-1 objects. For N spin-1/2 objects, the totally symmetric combination forms a unique $S_{\rm tot} = N/2$ (or fully-polarized ferromagnetic state) with a degeneracy of $2S_{\rm tot} + 1 = N + 1$ associated with different $S^z_{\rm tot}$ quantum numbers. For a spin-1 object, on the other hand, $S_{\rm tot}$ is no longer unique for the totally symmetric combination; it was found that [20]

$$S_{\rm tot} = N, N-2, N-4, \cdots, \tag{4}$$

with each value appearing exactly once. The total degeneracy is

$$D_0 = \sum_{S_{\rm tot}}(2S_{\rm tot} + 1) = (N+1)(N+2)/2. \tag{5}$$

An easier way to understand this larger degeneracy is to recognize that for each spin-1 object there are 3 internal states associated with $S^z = 0, \pm 1$; thus states formed by totally symmetric combinations of N spin-1 states form a *single* totally symmetric representation of SU(3) [20,21], which is represented by a row of N boxes in the Young tableaux or simply the representation $[N]$ [22]. The result, Eq. (4), may be viewed as decomposing a single irreducible representation of SU(3) into multiple irreducible representations of its subgroup SU(2).

The result (4) can also be obtained from an alternative method. The MR state can also be written as

$$\psi_{\rm MR} = \left[\prod_{i<j\le 2N}(z_i - z_j)\right] \times S\left[\prod_{0<i<j\le N}(z_i - z_j)^2 \prod_{N<k<l\le 2N}(z_k - z_l)^2\right], \tag{6}$$

where S is the symmetrizer. In Eq. (6) one divides the electrons into two groups, A and B; within each group one has the Jastrow factor $\prod_{0<i<j\le N}(z_i - z_j)^2$, which is then symmetrized among all particles. We now generalize Eq. (6) to include electron spins:

$$\psi(z_1, \chi_1; \cdots; z_{2N}, \chi_{2N}) = \left[\prod_{i<j\le 2N}(z_i - z_j)\right] S\left[\chi_A\chi_B\prod_{i<j\le N}(z_i - z_j)^2 \prod_{N<k<l\le 2N}(z_k - z_l)^2\right], \tag{7}$$

where χ_A and χ_B represent the spin-wave functions for clusters A and B respectively. The symmetrization imposes the following constraints on the spin-wave functions: (i) χ_A and χ_B are totally symmetric spin-wave functions of N spin-1/2 particles and thus each represents a spin-$N/2$ object; (ii) since the two clusters are also symmetrized, the total spin is a symmetric combination of two spin-$N/2$ objects, which leads to Eq. (4).

The construction above can be easily extended to the *RR* states [10] at level k to include spin, which are zero-energy states of a special $k+1$-body interaction:

$$\psi(z_1, \chi_1; \cdots; z_{kN}, \chi_{kN}) = \left[\prod_{i<j\le kN}(z_i - z_j)\right] S\left[\prod_{I=1}^{k}\left(\chi_I\prod_{0<i<j\le N}(z_i^I - z_j^I)^2\right)\right], \tag{8}$$

where we have divided $N_e = kN$ electrons into k clusters, χ_I is the spin-wave function of the Ith cluster, and z_i^I is the spatial coordinate of the ith electron of the Ith cluster. Using the same arguments as before, we find that we have k spin-$N/2$ objects (one from each cluster) forming totally symmetric combinations; the total ground state degeneracy is

$$D_0 = \frac{(k+N)!}{k!N!}, \tag{9}$$

which applies to the Laughlin ($k = 1$) and MR ($k = 2$) cases as well. It coincides with the totally symmetric $[N]$ representation of the SU$(k+1)$ group [22].

Our prediction of the spin quantum numbers for the case of $k = 2$ has been confirmed by exact diagonalization of the three-body Hamiltonian properly generalized to include spin degrees of freedom, on the sphere for up to 10 electrons. The Hamiltonian of Eq. (1) is not strictly positive definite when spin reversed states are included. In addition, it contains an arbitrary scale. We will work instead with a Hamiltonian made of projection operators:

$$\begin{aligned} H_{3B} = &\sum_{m=1}^{3} V_m P(3N_\phi/2 - m, 1/2) \\ &+ \sum_{n=1}^{2} V_4^n P_n(3N_\phi/2 - 4, 1/2) \\ &+ V_3' P(3N_\phi/2 - 3, 3/2) \\ &+ V_5 P(3N_\phi/2 - 5, 1/2), \end{aligned} \tag{10}$$

where N_ϕ is the total magnetic flux through the system and $P(L, S)$ projects out the state of angular momentum L and spin S. When such states are not unique we distinguish them with an index n. The V's are the three-body pseudopotential parameters [23] all of which were set to be 1. The projection operators P have unit eigenvalues as expected. The first 6 terms project out the states of 3 fermions with relative angular momentum less than 5, which are absent both in the MR state and the states of Eq. (3). The last term projects out all 3 fermionic states with relative angular momentum $m = 5$, and spin $S = 1/2$ [24] in which the opposite spins have relative angular momentum zero, which are also absent in Eq. (3).

Figure 1 shows the spectra for the cases of 8 electrons (4 pairs) and 10 electrons (5 pairs), respectively. We find that the ground states with zero energy all have total angular momentum $L = 0$, which is the same as the MR state, and their spin quantum numbers indeed take values $S_{\rm tot} = N, N-2, N-4, \cdots$, as predicted. In addition to ground states, the spectra of the lowest-energy excitations are also noteworthy. We see in both cases the lowest-energy excited states have total angular momentum $L = 1$ and have spin quantum numbers $S_{\rm tot} = 1, 2, \cdots, N-1$, with each multiplet appearing exactly once with nearly degenerate energies. If the degeneracy were exact, that would result in a total degeneracy for the first excited states

$$D_1 = \sum_{S_{\rm tot}=1}^{N-1} (2S_{\rm tot} + 1) = N^2 - 1. \tag{11}$$

In the following we argue that this can be understood as the consequence of an emergent SU(3) symmetry at low energies.

As discussed above, the ground states can be viewed as a single, totally symmetric SU(3) multiplet. If the system had an exact SU(3) symmetry, we could view the ground state as a fully magnetized SU(3) ferromagnet and the SU(3) symmetry would be spontaneously broken. Then the lowest-energy excitations of the system are expected to be SU(3) spin waves. The lowest-energy spin-wave state would have the smallest possible angular momentum $L = 1$ (corresponding to the smallest momentum in a translationally-invariant system) and has one SU(3) spin "flipped." In group theoretical language, a single "spin flip" means going from the totally symmetric representation $[N]$ to the representation $[N-1, 1]$ (which is represented by two rows in the Young tableaux with $N-1$ and 1 boxes, respectively), indicating one of the SU(3) spins is antisymmetrized with another. This mixed representation indeed has dimension $N^2 - 1$ [22], in agreement with Eq. (11), and it is easy to show that when decomposing this single SU(3) representation into SU(2) representations, one obtains $S_{\rm tot} = 1, 2, \cdots, N-1$. We thus conjecture the SU(3) symmetry is a property of the Hamiltonian (10) at low-energy; this is exact for the ground states, but for the excited states it is approximate, and supported by numerical evidence only. Should the symmetry become asymptotically exact in the long-distance limit, we would expect the degeneracy of the lowest-energy excited states to improve as system size increases and become asymptotically exact.

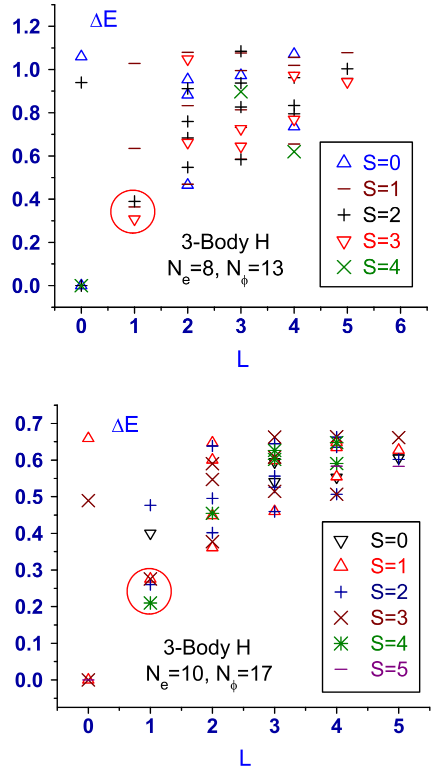

FIG. 1 (color online). Low-energy spectrum of the three-body Hamiltonian on the sphere at half-filling. The number of flux quanta N_Φ corresponds to a shift of 3, which is the same as that of the Moore-Read state. The ground states (at $L = 0$) with different total spin quantum number (S) form a single totally symmetric SU(3) multiplet corresponding to a fully-magnetized SU(3) ferromagnet; the low-energy excitations at $L = 1$ (inside red circle) are understood to be SU(3) spinwaves. Upper panel: System with 8 electrons (or 4 pairs); lower panel: 10 electrons or 5 pairs.

In a recent work [13], Dimov *et al.* argued that the low-energy effective theory of the ferromagnetic state at $\nu = 5/2$ is described by a perturbed CP^2 nonlinear σ model (NLσM). The original CP^2 NLσM possesses SU(3) symmetry; Dimov *et al.* argued that, for Coulomb or other generic two-body interactions, there exist symmetry-breaking perturbations in the effective theory that reduce the SU(3) symmetry down to SU(2), which is the symmetry possessed by the microscopic Hamiltonian. The 3-body Hamiltonian (10) also possesses SU(2) symmetry *only*. However our results suggest that for this very special case, the low-energy physics is very close to the original CP^2 NLσM with all the symmetry-breaking perturbations vanishing; in fact it may be possible to tune certain parameters in the Hamiltonian (10) to reach such a high symmetry point. If so, such a special three-body Hamiltonian would be a very useful point of departure for studying the various possible spin states and low-energy excitations above them at $\nu = 5/2$. If the ferromagnetic state at $\nu = 5/2$ indeed possesses approximate SU(3) symmetry, it will support *two* instead of just one low-energy spin-wave modes, and the skyrmions that appear when ν deviates from $5/2$ will have a richer spin structure [13]. Such differences from ordinary SU(2) quantum Hall ferromagnets can be probed using NMR and other experimental methods.

As emphasized earlier, the large spin degeneracies associated with the states described by Eqs. (3) and (8) are special properties of the special multiple-electron interaction Hamiltonians. For a generic Hamiltonian with SU(2) symmetry, the degeneracy between states in Eqs. (3) and (8) with different $S_{\rm tot}$ will be lifted. They will then represent quantum Hall states with different magnetization that varies essentially *continuously* from zero to full polarization. In general, one would expect these states to dominate the magnetization of the system at finite but low temperatures. Quasihole excitations on top of these ground states can be constructed in a manner similar to their spin-polarized counterparts; for example, a two-quasihole state on top of the ground state (3) with the same spin quantum number takes the form

$$\psi_{2\rm qh} = \left[\prod_{i<j}(z_i - z_j)^2\right] \times A\left[\left(\frac{(z_1-\eta_1)(z_2-\eta_2)+(z_1-\eta_2)(z_2-\eta_1)}{z_1-z_2}\cdots\right) \times\left(\sum_{\{\chi\}} c_{\{\chi\}}\chi_{12}\cdots\chi_{2N-1,2N}\right)\right], \qquad (12)$$

where η_1 and η_2 are the quasihole coordinates. Multi-quasi-hole states can be constructed similarly. Just like the quasihole states of the MR and RR states [25], the locations of the quasiholes do *not* uniquely determine the state when more than two quasiholes are present and the degeneracy grows exponentially with the quasihole number; these are thus *non-Abelian* quasiholes. Their braiding properties may also turn out to be the same as those of the MR state and will be left to future work. Another, but less likely [17], possibility would be a spontaneous breaking of the spin SU(2) symmetry that obtains the SU(3) degeneracy for generic Hamiltonians. If so, the quantum Hall state will be reduced to the 331 Abelian phase. The 331 state is *not* an eigenstate of $S_{\rm tot}$; it can be constructed as a linear superposition of states of the form (3) [26] with different $S_{\rm tot}$ but fixed $S^z_{\rm tot} = 0$.

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